Drohobych Ivan Franko State Pedagogical University

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SELECTED TOPICS OF THE THEORY OF FUNCTIONS

Lecture texts, practical and individual tasks

Educational and methodical manual for students of specializations
014 «Secondary Education (Mathematics) » and 111 «Mathematics»

Drohobych
2025

Дрогобицький державний педагогічний університет імені Івана Франка

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ВИБРАНІ РОЗДІЛИ ТЕОРІЇ ФУНКЦІЙ

Тексти лекцій, практичні та індивідуальні завдання

Навчально-методичний посібник
для студентів спеціальностей
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The educational and methodological manual is written in accordance with the curriculum of the academic discipline "Selected topics of the theory of functions" for the training of specialists at the second (master's) level of higher education in the specialties 014 «Secondary Education (Mathematics) » and 111 «Mathematics» within the fields of knowledge 01 «Education/Pedagogy» and 11 «Mathematics and Statistics». The program was approved by the Academic Council of Drohobych Ivan Franko State Pedagogical University. The manual includes lecture texts, tasks for practical classes, individual tasks, tasks for independent work, as well as materials for ongoing and final assessment.

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Рекомендовано до друку вченою радою Дрогобицького державного педагогічного університету імені Івана Франка (протокол № 2 від 27 лютого 2025 р.)

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Х 28 **Вибрані розділи теорії функцій :** тексти лекцій, практичні та індивідуальні завдання : навч.-метод. пос. Дрогобич : Дрогобицький державний педагогічний університет імені Івана Франка, 2025. 177 с.

Навчально-методичний посібник написано відповідно до програми навчальної дисципліни „Вибрані розділи теорії функцій” для підготовки фахівців другого (магістерського) рівня вищої освіти за спеціальностями 014 «Середня освіта (Математика)», 111 «Математика» галузей знань 01 «Освіта/Педагогіка» та 11 «Математика та статистика», затвердженої вченою радою Дрогобицького державного педагогічного університету імені Івана Франка. Він містить лекційний матеріал, завдання для практичних занять, індивідуальні завдання, завдання для самостійної роботи, поточного та підсумкового контролю.

Бібліографія: 57 назв

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Basic notations

1. $\mathbb{N}$ is the set of all natural numbers.
2. $\mathbb{Z}$ is the set of all integers.
3. $\mathbb{N}_0$ or $\mathbb{Z}_+$ is the set of all non-negative integers.
4. $\overline{n;m}$ is the set of all integers x satisfying the inequality $n \le x \le m$.
5. $\mathbb{Q}$ is the set of all rational numbers.
6. $\mathbb{R}$ is the set of all real numbers.
7. $\overline{\mathbb{R}}$ is the set of all real numbers extended by the symbols "$-\infty$" and "$+\infty$".
8. $\overline{\mathbb{R}}_0$ is the set of all real numbers extended by the symbol "∞".
9. $\mathbb{C}$ is the set of all complex numbers.
10. $(a;b)$ is an open interval, i.e., the set of all real numbers x satisfying the inequality $a < x < b$.
11. $[a;b]$ is a closed interval, i.e., the set of all real numbers x satisfying the inequality $a \le x \le b$.
12. $(a;b]$ is a half-open interval with the right endpoint included, i.e., the set of all real numbers x satisfying $a < x \le b$.
13. $[a;b)$ is a half-open interval with the left endpoint included, i.e., the set of all real numbers x satisfying $a \le x < b$.
14. $D(f)$ is the domain of a function $f : H_1 \to H_2$.
15. $E(f)$ is the range of a function $f : H_1 \to H_2$.
16. $U(a;\varepsilon)$ is the ε-neighborhood of the point a.
17. $\overset{\circ}{U}(a;\varepsilon)$ is the punctured ε-neighborhood of the point a.
18. $\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-ixt} dt$ is the Fourier transform of a function $f : \mathbb{R} \to \mathbb{C}$.
19. $\check{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{itx} dx$ is the inverse Fourier transform of a function $f : \mathbb{R} \to \mathbb{C}$.
20. $f * \varphi(x) = \int_{-\infty}^{+\infty} f(x-\tau)\varphi(\tau) d\tau$ is the convolution of two functions $f : \mathbb{R} \to \mathbb{C}$ and $\varphi : \mathbb{R} \to \mathbb{C}$.

21. $M_f(r)=\max\{|f(z)|:|z|\le r\}$ is the maximum of the modulus of an entire function f.

22. $\mu_f(r)=\max\{|f_k|r^k : k\ge 0\}$ is the maximal term of an entire function f.

23. $\nu_f(r)=\{k:\mu_f(r)=|f_k|r^k\}$ is the central index of an entire function f.

24. $n(t)=\sum\limits_{|\lambda_k|\le t} 1=\max\{k:|\lambda_k|\le t\}$ is the counting function of the sequence (λ_k).

25. $N(r)=\int\limits_0^r \frac{n(t)-n(0)}{t}dt+n(0)\ln r$ is the averaged counting function of the sequence (λ_k).

26. $E(w;p)=\begin{cases}1-w, & p=0,\\ (1-w)\exp\left(\sum\limits_{k=1}^{p} w^k/k\right), & p\in\mathbb{N},\end{cases}$ is the Weierstrass primary factor.

27. $h_f(\theta)=\overline{\lim\limits_{r\to+\infty}}\frac{\ln|f(re^{i\theta})|}{r^\rho}$ is the indicator function of an entire function f of order $\rho\in(0;+\infty)$.

28. $\operatorname{conv} D$ is the convex hull of a set $D\subset\mathbb{C}$.

29. $k_D(\theta)=\sup\{\operatorname{Re}(ze^{-i\theta}):z\in D\}$ is the supporting function of a set $D\subset\mathbb{C}$.

30. $\gamma_L(z)=\sum\limits_{n=0}^{\infty}\frac{n!L_n}{z^{n+1}}$ is the Borel transform of a function $L(z)=\prod\limits_{n=0}^{\infty} L_n z^n$.

31. PW_σ^2 is the set of all entire functions of exponential type $\le\sigma\in(0;+\infty)$, whose narrowing on $\mathbb{R}$ belongs to the space $L_2(\mathbb{R})$.

32. $H^p(\mathbb{C}_+)$ is the class of functions holomorphic in the half-plane $\mathbb{C}_+=\{z=x+iy:x>0\}$, for which $\|f\|:=\sup\{|f(x+iy)|:x>0\}<+\infty$, if $p=+\infty$, and $\|f\|^p:=\sup\left\{\int\limits_{-\infty}^{+\infty}|f(x+iy)|^p\,dy:x>0\right\}<+\infty$ for $p\in[1;+\infty)$.

33. $S(\mathbb{R})$ is the space of rapidly decreasing functions.

34. $C^{(\infty)}(\mathbb{R})$ is the set of infinitely differentiable functions $\varphi:\mathbb{R}\to\mathbb{C}$ on $\mathbb{R}$.
35. $C_0^{(\infty)}(\mathbb{R})$ is the set of all infinitely differentiable functions $\varphi:\mathbb{R}\to\mathbb{C}$ with compact support.
36. $(C_0^{(\infty)}(\mathbb{R}))'$ is the space of generalized functions.
37. $L_{1,loc}(\mathbb{R})$ is the space of all locally integrable functions on $\mathbb{R}$.
38. $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ is the Laplace operator.
39. ► is the end of a proof.
40. $:=$ is defined as.

Preface

The ability to perform asymptotic estimates and investigate the basic properties of various classes of functions is fundamental to the success of a mathematician conducting research in any branch of mathematics, in particular, in the theory of functions. The theory of functions has a multitude of applications in calculus and functional analysis. This educational and methodical manual explores the simplest methods for obtaining asymptotic estimates of sum and integrals, finding the asymptotics of inverse functions and roots of equations, the main properties of entire and subharmonic functions, generalized functions and some other classes of functions. Many sections of this manual contain problems with applications related to these topics.

The proposed guide is a revised and expanded edition of the manual [2]. It includes lecture material, tasks for practical classes, individual tasks, tasks for independent work, and modular control tasks. Key theoretical facts are accompanied by solutions to a large number of typical examples.

To understand the text of the manual, you need to know the basics of conventioanal courses on mathematical and complex analysis.

Designed for students specializing in 014 “Secondary Education (Mathematics)” and 111 “Mathematics”.

Dedicated to the memory of Professor Bohdan V. Vynnyts’kyi

Chapter 1. Elementary asymptotic methods

1.1. O-symbolism. Landau symbols. Often, when considering both theoretical and applied problems, it is necessary to compare one complex function f with another or replace it with a simpler function φ on a certain set E or in some neighborhood of a given point a, in order to obtain a clear mathematical description of the corresponding problem. In this process, the question arises as to which functions should be considered close. The answer to this depends on the problem being considered.

Example 1. *For large $n \in \mathbb{N}$, the functions*

$$f(t) = \begin{cases} n, & t \in [0;1/n^3], \\ t, & t \in [1/n^3;2], \end{cases}$$

and $\varphi(t) = t$ are close on the segment $[0;2]$ in the mean square sense, because

$$d(f;\varphi) := \| f - \varphi \| := \left(\int_0^2 | f(t) - \varphi(t) |^2 \, dt \right)^{1/2} = \left(\int_0^{1/n^3} (n-t)^2 \, dt \right)^{1/2} =$$

$$= \frac{(n - 1/n^3)^3 - n^3}{-3} \to 0.$$

However, it is difficult to call them close in the $\sup$*-norm, because*

$$d(f;\varphi) := \| f - \varphi \| := \sup\{ | f(t) - \varphi(t) | : t \in [0;2] \} = n \to +\infty .$$

If a function f in the neighborhood of a point a is replaced by a simpler function φ, then $| f(x) - \varphi(x) |$ and $|(f(x) - \varphi(x)) / f(x) |$ are called the absolute and relative errors, respectively. In solving such problems, certain symbols (Landau symbols) are often used [4, 16, 27-29, 48, 53]:

1. The symbol “ $f(x) = o(1)$, $x \to a$ ” means that $\lim\limits_{x \to a} f(x) = 0$.

2. The symbol “ $f(x) = o\big(\varphi(x)\big)$, $x \to a$ ” means that $\lim\limits_{x \to a} f(x)/\varphi(x) = 0$, i.e., $f(x)/\varphi(x) = o(1)$ as $x \to a$.

3. The symbol “ $f(x) = O(1)$, $x \to a$ ” means that the function f is bounded in some punctured neighborhood of a point a.

4. The symbol “ $f(x) = O\big(\varphi(x)\big)$, $x \to a$ ” means that the function $f(x)/\varphi(x)$ is bounded in some punctured neighborhood of a point a, i.e., $f(x)/\varphi(x) = O(1)$ as $x \to a$.

5. The symbol “ $f(x) \asymp \varphi(x)$, $x \to a$ ” means that $f(x) = O\big(\varphi(x)\big)$ as $x \to a$ and $\varphi(x) = O\big(f(x)\big)$ as $x \to a$.

6. The symbol “ $f(x) = O(1)$, $x \in E$ ” means that the function f is bounded on a set E.

7. The symbol “ $f(x) = O\big(\varphi(x)\big)$, $x \in E$ ” means that $f(x)/\varphi(x) = O(1)$ as $x \in E$.

8. The symbol “ $f(x) \asymp \varphi(x)$, $x \in E$ ” means that $f(x) = O\big(\varphi(x)\big)$ as $x \in E$, and $\varphi(x) = O\big(f(x)\big)$ as $x \in E$.

9. The symbol “ $f(x) \sim \varphi(x)$, $x \to a$ ” means that $\lim\limits_{x \to a} f(x)/\varphi(x) = 1$.

Equalities that involve Landau symbols are called asymptotic formulas or asymptotic equalities. In such formulas, the symbol $o\big(\varphi(x)\big)$ denotes any function f from the considered class for which $f(x) = o\big(\varphi(x)\big)$ as $x \to a$, or some specific such function [4, 16, 27-29, 48, 53]. A similar interpretation applies to other symbols being considered. Thus, these symbols should be read from left to right.

Example 2. $\operatorname{tg} x = o(1)$, $x \to 0$; $\sin x \sim x$, $x \to 0$; $\cos x = O(1)$, $x \in \mathbb{R}$.

Example 3. *$o(1) + o(1) = o(1)$ as $x \to a$ (sum of two infinitesimally small quantities is infinitesimally small).*

Example 4. *$O(1) \cdot o(1) = o(1)$ as $x \to a$ (product of a bounded function and an infinitesimally small function is infinitesimally small).*

Example 5. $(1 + o(1))(1 + o(1)) = 1 + o(1)$, $x \to a$.

Example 6. $\dfrac{1}{1 + o(1)} = 1 + o(1)$, $x \to a$.

Example 7. $O(x^2) + o(x) = o(x) = o(1)$, $x \to 0$.

Example 8.

$$x + O(x^3) + x^2 + o(x^2) = x + x^2 + o(x^2) =$$
$$= x + O(x^2) = x + o(x) = x(1 + o(1)),\ x \to 0.$$

Example 9.

$$x^4 + O(x) + x^2 + o(x^2) = x^4 + x^2 + o(x^2) =$$
$$= x^4 + O(x^2) = x^4 + o(x^4) = x^4(1 + o(1)),\ x \to \infty.$$

Example 10. *If $\varlimsup\limits_{x \to a} \big| f(x)/\varphi(x) \big| < +\infty$, then $f(x) = O\big(\varphi(x)\big)$ as $x \to a$.*

Example 11. *If* $\overline{\lim\limits_{x\to a}}\left|f(x)/\varphi(x)\right|<+\infty$, *then* $f(x)=O\big(\varphi(x)\big)$ *as* $x\to a$.

Example 12. *If* $0<\underline{\lim\limits_{x\to a}}\left|f(x)/\varphi(x)\right|\le\overline{\lim\limits_{x\to a}}\left|f(x)/\varphi(x)\right|<+\infty$, *then* $f(x)\asymp\varphi(x)$ *as* $x\to a$.

Remark 1. *Sometimes, the considered symbols are used in a slightly more general sense.*

If the functions f and φ are infinitesimal at the point a, and $f(x)=o\big(\varphi(x)\big)$ as $x\to a$, then the infinitesimal f is said to be of a higher order than the infinitesimal φ [4, 16, 27-29, 48, 53]. If the functions f and φ are infinitesimal at the point a, and $f(x)\asymp\varphi(x)$ as $x\to a$, then the infinitesimals f and φ are said to be of the same order at the point a. If the functions f and φ are infinitesimal at the point a, and $f(x)\sim\varphi(x)$ as $x\to a$, then the infinitesimals f and φ are said to be equivalent [4, 16, 27-29, 48, 53].

Example 13. *An infinitesimal function* $f(x)=x^5$ *at the point* $a=0$ *is of a higher order than the infinitesimal function* $\varphi(x)=x^4$ *at the same point because* $x^5=o(x^4)$ *as* $x\to 0$.

Example 14. *Infinitesimal functions* $f(x)=x^2$ *and* $\varphi(x)=\sin 3x^2$ *at the point* $a=0$ *are of the same order.*

Example 15. *Infinitesimal functions* $f(x)=x^3$ *and* $\varphi(x)=\operatorname{tg}^3 x$ *at the point* $a=0$ *are equivalent.*

Example 16. *Infinitesimal functions*

$$\varphi(x)=\begin{cases} x, & x\in(0;+\infty),\\ x^2, & x\in(-\infty;0),\end{cases}\quad \textit{and}\quad \varphi(x)=\begin{cases} x, & x\in(0;+\infty),\\ x^2, & x\in(-\infty;0),\end{cases}$$

at the point $a=0$ *are incomparable in the sense that neither of them is of a higher order than the other, they are not equivalent, and they are not of the same order.*

If the functions f and φ are infinitely large at the point a, and $f(x)=o\big(\varphi(x)\big)$ as $x\to a$, then the infinitely large function φ is said to be of a higher order than the infinitely large function f [4, 16, 27-29, 48, 53].

Example 17. *An infinitely large function* $\varphi(x)=x^4$ *at the point* $a=\infty$ *is of a higher order than the infinitely large function* $f(x)=x^3$ *at the same point because* $x^3=o(x^4)$ *as* $x\to\infty$.

Example 18. *An infinitely large function* $\varphi(x)=e^{2x}$ *at the point* $a=+\infty$ *is of a higher order than the infinitely large function* $f(x)=x^2$ *at the same point because* $x^2=o(e^{2x})$ *as* $x\to+\infty$.

1.2. Application of Taylor's formula with remainder in Peano's form for calculating limits and asymptotic formulas. If a function $f:\mathbb{R}\to\mathbb{R}$ has a derivative of order $n\in\mathbb{Z}_+$ at the point $a\in\mathbb{R}$, then the Taylor formula with the remainder term in the Peano form can be expressed as [4, 16, 27-29, 48, 53]:

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k+o((x-a)^n)=$$
$$=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k+O((x-a)^{n+1}),\ x\to a.$$

If all the terms $f^{(k)}(a)\neq 0$ are infinitesimal at the point a, then in the sum $\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k$ each subsequent term is of a higher order of infinitesimal at the point a than the previous one. This formula is conveniently used for finding limits and deriving various asymptotic formulas.

Example 1. *We have the following expansions*

$$e^x=\sum_{k=0}^{n}\frac{1}{k!}x^k+o(x^n)=\sum_{k=0}^{n}\frac{1}{k!}x^k+O(x^{n+1}),\ x\to 0,$$
$$e^{-x}=\sum_{k=0}^{n}\frac{(-1)^k}{k!}x^k+o(x^n)=\sum_{k=0}^{n}\frac{(-1)^k}{k!}x^k+O(x^{n+1}),\ x\to 0,$$

for every $n\in\mathbb{Z}_+$.

Example 2. *The following asymptotic equalities*

$$\sin x=\sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)!}x^{2k+1}+o(x^{2n+2})=\sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)!}x^{2k+1}+O(x^{2n+3}),\ x\to 0,$$
$$\cos x=\sum_{k=0}^{n}\frac{(-1)^k}{(2k)!}x^{2k}+o(x^{2n+1})=\sum_{k=0}^{n}\frac{(-1)^k}{(2k)!}x^{2k}+O(x^{2n+2}),\ x\to 0,$$

are valid for any $n\in\mathbb{Z}_+$.

Example 3. *We have the following expansions*

$$\operatorname{sh} x=\sum_{k=0}^{n}\frac{1}{(2k+1)!}x^{2k+1}+o(x^{2n+2})=\sum_{k=0}^{n}\frac{1}{(2k+1)!}x^{2k+1}+O(x^{2n+3}),\ x\to 0,$$

$$\operatorname{ch} x=\sum_{k=0}^{n}\frac{1}{(2k)!}x^{2k}+o(x^{2n+1})=\sum_{k=0}^{n}\frac{1}{(2k)!}x^{2k}+O(x^{2n+2}),\ x\to 0,$$

for every $n\in\mathbb{Z}_+$.

Example 4. *The following asymptotic formulas*

$$\ln(1+x)=\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}x^{k}+o(x^{n})=\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}x^{k}+O(x^{n+1}),\ x\to 0,$$

$$\ln(1-x)=-\sum_{k=1}^{n}\frac{1}{k}x^{k}+o(x^{n})=-\sum_{k=1}^{n}\frac{1}{k}x^{k}+O(x^{n+1}),\ x\to 0,$$

are valid for any $n\in\mathbb{Z}_+$.

Example 5. *We have the expansion*

$$(1+x)^{\alpha}=1+\sum_{k=1}^{n}\frac{\prod_{j=1}^{k}(\alpha-j+1)}{k}x^{k}+o(x^{n})=$$

$$=1+\sum_{k=1}^{n}\frac{\prod_{j=1}^{k}(\alpha-j+1)}{k}x^{k}+O(x^{n+1}),\ x\to 0,$$

for every $n\in\mathbb{Z}_+$ *and* $\alpha\in\mathbb{R}$, *where* $\prod_{j=1}^{0}(\alpha-j+1):=1$. *In particular,*

$$\frac{1}{1-x}=\sum_{k=0}^{n}x^{k}+o(x^{n})=\sum_{k=0}^{n}x^{k}+O(x^{n+1}),\ x\to 0,$$

$$\frac{1}{1+x}=\sum_{k=0}^{n}(-1)^{k}x^{k}+o(x^{n})=\sum_{k=0}^{n}(-1)^{k}x^{k}+O(x^{n+1}),\ x\to 0,$$

for every $n\in\mathbb{Z}_+$.

Example 6. *We can write the following asymptotic formula as* $x\to 0$:

$$\sin x-\frac{x}{1+x}=x-\frac{x^3}{6}+o(x^3)-x(1-x+x^2+o(x^2))=$$

$$=-\frac{7x^3}{6}+x^2+o(x^3).$$

Example 7. *We derive the following asymptotic formula as* $x\to 0$:

$$\ln(1-x)-\cos x+O(x^3)+o(x^2)=$$

$$=-x-\frac{x^2}{2}+o(x^2)-1+\frac{x^2}{2}+o(x^3)+O(x^3)+o(x^2)=$$

$$=-x-1+o(x^3)+O(x^3)+o(x^2)=-1-x+o(x^2).$$

Example 8. *We have the following asymptotic formula as* $x\to 0$:

$$\ln(1+x+o(x^2))=x+o(x^2)-\frac{1}{2}\left(x+o(x^2)\right)^2+o\left(\left(x+o(x^2)\right)^2\right)=$$

$$=x-\frac{1}{2}x^2+o(x^2).$$

Example 9. *If* $f(x)=1-x+o(x)$ *and* $\varphi(x)=1-2x+o(x)$ *as* $x\to 0$, *then* $f(\varphi(x))=2x+o(x)+o(1-2x+o(x))=2x+o(1)$ *as* $x\to 0$.

Example 10. *If* $m\in\mathbb{R}$, *then* $(1+x)^m=1+mx+o(x)$ *as* $x\to 0$, *and*

$$(1+x)^m-x^m=x^m\left(\left(1+\frac{1}{x}\right)^m-1\right)=$$

$$=x^m\left(1+\frac{m}{x}+o\left(\frac{1}{x}\right)-1\right)=mx^{m-1}+o(x^{m-1}),\ x\to+\infty.$$

When finding limits using the Taylor formula [4, 16, 27-29, 48, 53], it is important to choose the order n wisely. It is preferable to choose n as small as possible (if a certain n works in a given situation, any n larger than it will also work, but not the other way around).

Example 11. *For finding the limit*

$$\lim_{x\to 0}\frac{\sin x-x}{x^3}$$

we will use the Taylor formula

$$\sin x=\sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)!}x^{2k+1}+o(x^{2n+1}),\ x\to 0,$$

taking $n=1$. *Then*

$$\lim_{x\to 0}\frac{\sin x-x}{x^3}=\lim_{x\to 0}\frac{x-\frac{x^3}{6}+o(x^3)-x}{x^3}=$$

$$=\lim_{x\to 0}\frac{-\frac{x^3}{6}+o(x^3)}{x^3}=\lim_{x\to 0}\left(-\frac{1}{6}+o(1)\right)=-\frac{1}{6}.$$

If we take $n=2$, *then we again obtain the desired result:*

$$\lim_{x\to 0}\frac{\sin x - x}{x^3} = \lim_{x\to 0}\frac{x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5) - x}{x^3} =$$

$$\lim_{x\to 0}\frac{-\frac{x^3}{6} + \frac{x^5}{120} + o(x^5)}{x^3} = \lim_{x\to 0}\left(-\frac{1}{6} + \frac{x^2}{120} + o(x^2)\right) = -\frac{1}{6}.$$

Example 12. *Since*

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3),\ x\to 0,$$

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + o(x^3),\ x\to 0,$$

$$\sin x = x - \frac{x^3}{6} + o(x^3),\ x\to 0,$$

we obtain

$$\lim_{x\to 0}\frac{e^x - e^{-x} - 2x}{x - \sin x} = \lim_{x\to 0}\frac{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3) - \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + o(x^3)\right) - 2x}{x - \left(x - \frac{x^3}{6} + o(x^3)\right)} =$$

$$= \lim_{x\to 0}\frac{\frac{x^3}{3} + o(x^3)}{\frac{x^3}{6} + o(x^3)} = \lim_{x\to 0}\frac{\frac{1}{3} + o(1)}{\frac{1}{6} + o(1)} = 2.$$

Example 13. *Since*

$$\lim_{x\to 0}\frac{1}{x}\ln\frac{\sin x}{x} = \lim_{x\to 0}\frac{\ln\frac{x - \frac{x^3}{6} + o(x^3)}{x}}{x} =$$

$$= \lim_{x\to 0}\frac{\ln\left(1 - \frac{x^2}{6} + o(x^2)\right)}{x} = \lim_{x\to 0}\frac{-\frac{x^2}{6} + o(x^2)}{x} = -\lim_{x\to 0} x\left(\frac{1}{6} + o(1)\right) = 0,$$

we have

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x}} = e^{\lim_{x\to 0}\frac{1}{x}\ln\frac{\sin x}{x}} = 1.$$

When finding limits, the following Stolz theorem is often useful [4, 16, 27-29, 48, 53].

Example 14 (Stolz theorem). *For any two sequences* (x_n) *and* (y_n) *such that* $\lim\limits_{n\to\infty} y_n = +\infty$ *and* $y_n - y_{n-1} > 0$ *for all* $n \geq n^*$, *holds:*

$$\varliminf_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} \leq \varliminf_{n\to\infty} \frac{x_n}{y_n} \leq \varlimsup_{n\to\infty} \frac{x_n}{y_n} \leq \varlimsup_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} .$$

Example 15. *For any sequence* (u_n) *that converges in* $\overline{\mathbb{R}}_0$, *we have*

$$\lim_{n\to\infty} \frac{u_1 + u_2 + ... + u_n}{n} = \lim_{n\to\infty} u_n .$$

Indeed, if $x_n = \sum\limits_{k=1}^{n} u_k$ *and* $y_n = n$, *then*

$$\frac{x_n}{y_n} = \frac{u_1 + u_2 + ... + u_n}{n}, \quad \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = u_n ,$$

and by Stolz's theorem, we get

$$\lim_{n\to\infty} \frac{u_1 + u_2 + ... + u_n}{n} = \lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = \lim_{n\to\infty} u_n .$$

Example 16. *If* $x_n = \sum\limits_{k=1}^{n} k^3$ *and* $y_n = n^4$, *then by Stolz's theorem, we obtain*

$$\lim_{n\to\infty} \frac{\sum\limits_{k=1}^{n} k^3}{n^4} = \lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = \lim_{n\to\infty} \frac{n^3}{n^4 - (n-1)^4} = \frac{1}{4} .$$

1.3. Order and type of a function. Let $\eta : [0;+\infty) \to (0;+\infty)$ be a non-decreasing function. The order of a function η is defined as the number $\rho = \rho[\eta]$, determined by the formula [16, 27-29, 48, 53]:

$$\rho = \varlimsup_{t\to+\infty} \frac{\ln \eta(t)}{\ln t} .$$

In other words, the order of a function η is called the exact lower bound of those numbers $\rho_1 \in (0;+\infty]$, for which $(\exists c_1)(\forall t \in [1;+\infty)) : \eta(t) \leq c_1 t^{\rho_1}$), that is, $(\exists t_0)(\forall t \in [t_0;+\infty)) : \eta(t) \leq t^{\rho_1}$. The order of a function η is zero if and only if [16, 27-29, 48, 53]

$$(\forall \rho_1 \in (0;+\infty))(\exists t_0 \in (0;+\infty))(\forall t \geq t_0) : \eta(t) \leq t^{\rho_1} .$$

The order of a function η is $+\infty$ if and only if there exists a sequence (t_k),

$0 < t_k \uparrow +\infty$, such that [16, 27-29, 48, 53]

$$(\forall \rho_1 \in (0;+\infty))(\exists k_0 \in \mathbb{N})(\forall k \geq k_0) : \eta(t_k) \geq t_k^{\rho_1} .$$

The order of a function η is equal to a number $\rho \in (0;+\infty)$ if and only if two conditions are satisfied: 1) $(\forall \rho_1 > \rho)(\exists t_0 \in (0;+\infty))(\forall t \geq t_0) : \eta(t) \leq t^{\rho_1}$; 2) there exists a sequence (t_k), $0 < t_k \uparrow +\infty$, such that [16, 27-29, 48, 53]

$$(\forall \rho_2 < \rho)(\exists k_0 \in \mathbb{N})(\forall k \geq k_0) : \eta(t_k) \geq t_k^{\rho_2} .$$

If $\rho \in (0;+\infty)$ is the order of a function η, then the number $\sigma = \sigma[\eta] = \sigma[\eta;\rho]$, defined by the formula [16, 27-29, 48, 53]

$$\sigma = \overline{\lim_{t\to+\infty}} \frac{\eta(r)}{r^{\rho}},$$

is called the type of a function η with respect to the order ρ. The type of a function η with respect to the order $\rho \in (0;+\infty)$ is equal to zero if and only if

$$(\forall \sigma_1 \in (0;+\infty))(\exists t_0 \in (0;+\infty))(\forall t \geq t_0) : \eta(t) \leq \sigma_1 t^{\rho} .$$

Example 1. *Let* $\rho \in (0;+\infty)$, $\beta \in (0;+\infty)$, $\alpha \in (0;+\infty)$ *and* $\eta(t) = \alpha t^{\beta} + \ln(t+1)$. *Then* $\eta(t) = \alpha t^{\beta}(1+o(1))$ *and* $\ln \eta(t) = \beta(1+o(1))\ln t$ *as* $t \to +\infty$. *Therefore,* $\rho[\eta] = \beta$ *and* $\sigma[\eta] = \alpha$.

Example 2. *Let* $\eta(t) = t^{\beta}\ln(t+1)$, *where* $\beta \in (0;+\infty)$. *Then* $\ln \eta(t) = \beta(1+o(1))\ln t$ *as* $t \to +\infty$. *Therefore,* $\rho[\eta] = \beta$ *and* $\sigma[\eta] = +\infty$.

Example 3. *If* $\eta(t) = e^{\ln^2 t}$, *then* $\ln \eta(t) = \ln^2 t$. *Therefore,* $\rho[\eta] = +\infty$ *and* $\sigma[\eta] = +\infty$.

Example 4. *If the function* $\eta : [0;+\infty) \to (0;+\infty)$ *is continuous on* $[0;+\infty)$, *and for some* $\beta > -\rho$ *the function* $t^{\beta}\eta(t)$ *is non-decreasing on* $[1;+\infty)$ *and* $\int_1^{+\infty} t^{-\rho-1}\eta(t) < +\infty$, *then* $\eta(t) = o(t^{\rho})$ *as* $t \to +\infty$. *Indeed, we have*

$$0 \leftarrow \int_x^{+\infty} \frac{\eta(t)}{t^{\rho+1}} dt \geq x^{\beta}\eta(x) \int_x^{+\infty} \frac{1}{t^{\rho+\beta+1}} dt \geq \frac{\eta(x)}{(\rho+\beta)x^{\rho}}, \ x \to +\infty .$$

***Example 5* ([16, 27-29, 48, 53]).** *Let the function* $\eta : [0;+\infty) \to (0;+\infty)$ *is non-decreasing on* $[1;+\infty)$ *and* $f(x) = \int_1^x \eta(t) d\ln t$. *Then*

$$f(x)=\int_1^x \frac{\eta(t)}{t}dt \le \eta(x)\ln x,\ x\in[1;+\infty),$$

$$f(x)=\int_{x/e}^x \frac{\eta(t)}{t}dt \ge \eta(x/e),\ x\in[1;+\infty).$$

Therefore, $\ln f(x)\le \ln\eta(x)+\ln\ln x$ *and* $\rho[f]\le\rho[\eta]$. *In addition,*

$$\frac{\ln f(x)}{\ln x}\ge\frac{\ln\eta(x/e)}{\ln(x/e)}\cdot\frac{\ln(x/e)}{\ln x}$$

and $\rho[f]\ge\rho[\eta]$. *Hence,* $\rho[f]=\rho[\eta]$. *Further,*

$$\frac{f(x)}{x^\rho}\ge\frac{\eta(x/e)}{(x/e)^\rho}e^{-\rho}$$

and $\sigma[\eta;\rho]e^{-\rho}\le\sigma[f;\rho]$. *Furtermore, if* $\sigma[\eta;\rho]<+\infty$, *then* $t^{-\rho}\eta(t)\le\sigma_1$ *for each* $\sigma_1>\sigma[\eta;\rho]$ *and all* $t\ge t_0$. *Thus,*

$$f(x)=\int_{t_0}^x \frac{\eta(t)}{t}dt+O(1)\le\sigma_1\int_{t_0}^x t^{\rho-1}dt+O(1)=\frac{\sigma_1 x^\rho}{\rho}+O(1),\ x\ge t_0.$$

Hence, $\sigma[\eta;\rho]e^{-\rho}\le\sigma[f;\rho]\le\sigma[\eta;\rho]/\rho$.

1.4. Slowly varying functions. A function $\omega:\mathbb{R}\to\mathbb{R}$ is called slowly varying if [16, 27-29, 48, 53]

$$\lim_{t\to+\infty}\frac{\omega(c_1t)}{\omega(t)}=1$$

for every $c_1>0$, moreover the limit exists uniformly with respect to c_1 on every interval $[a;b]\subset\mathbb{R}$. In addition, if the function ω is non-decreasing, then it is called a slowly increasing function.

Example 1. *The function* $\omega(t)=\ln t$ *is slowly varying because*

$$\frac{\ln c_1t}{\ln t}=\frac{\ln c_1+\ln t}{\ln t}\to 1 \text{ as } t\to+\infty$$

uniformly with respect to $c_1\in[a;b]$. *The functions* $\omega(t)=\ln^2 t$, $\omega(t)=\ln\ln t$, $\omega(t)=\operatorname{arctg} t$, $\omega(t)=e^{1/t}$ *and* $\omega(t)=e^{\ln^\alpha t}$, *where* $\alpha\in[0;1)$, *are also slowly varying. The functions* $\omega(t)=t$, $\omega(t)=\sqrt{t}$, $\omega(t)=1/t^2$, $\omega(t)=e^t$, $\omega(t)=e^{\sqrt{t}}$ *and* $\omega(t)=e^{\ln^2 t}$, *are not slowly varying.*

Example 2. *Let the function* $\omega:\mathbb{R}\to\mathbb{R}$ *is differentiable on* $[0;+\infty)$ *and*

$$\lim_{t\to+\infty} t\omega'(t)/\omega(t)=0. \qquad (1)$$

Then ω is a slowly varying function because, by Lagrange's theorem, we obtain

$$\ln\omega(c_1t)-\ln\omega(t)=\frac{\xi\omega'(\xi)}{\omega(\xi)}\frac{(c_1-1)t}{\xi}\to 0,\ t\to+\infty,$$

uniformly in $c_1\in[a;b]$, where ξ lies between t and c_1t.

Example 3. *If the function $\omega:\mathbb{R}\to\mathbb{R}$ is continuously differentiable on $[0;+\infty)$ and satisfies condition (1), then $\lim_{t\to+\infty}\frac{\ln\omega(t)}{\ln t}=0$, because*

$$\lim_{x\to+\infty}\frac{\ln\omega(x)}{\ln x}=\lim_{x\to+\infty}\left(\frac{\ln\omega(0)}{\ln x}+\frac{\int_0^x\frac{\omega'(t)}{\omega(t)}dt}{\ln x}\right)=\lim_{x\to+\infty}x\frac{\eta'(x)}{\eta(x)}=0.$$

The class of slowly varying functions coincides with the set of functions ω that can be represented in the form [16, 27-29, 48, 53]

$$\omega(x)=\exp\left\{\omega_0(x)+\int_{x_0}^x\frac{w_0(t)}{t}dt\right\},$$

where $x_0\in\mathbb{R}$, ω_0 is a bounded measurable function, w_0 is a continuous function, and

$$\lim_{x\to+\infty}w_0(x)=0,\ \lim_{x\to+\infty}\omega_0(x)=c\in\mathbb{R}.$$

Example 4. *A positive measurable function $\omega:\mathbb{R}\to\mathbb{R}$ is slowly varying if and only if*

$$(\forall\delta\in(0;1))(\forall\alpha>1)(\exists t_0)(\forall t_1\ge t_0)(\forall t_2\ge t_1):$$

$$\frac{1}{\alpha}\omega(t_1)(t_2/t_1)^{-\delta}\le\omega(t_2)\le\alpha\omega(t_1)(t_2/t_1)^{\delta}.$$

Example 5. *A non-decreasing positive function $\omega:\mathbb{R}\to\mathbb{R}$ is slowly varying if and only if*

$$(\forall\delta\in(0;1))(\forall\alpha>1)(\exists t_0)(\forall t_1\ge t_0)(\forall t_2\ge t_1):\omega(t_2)\le\alpha\omega(t_1)(t_2/t_1)^{\delta}.$$

1.5. Proximate order. A proximate order is called [16, 27-29, 48, 53] a continuously differentiable function $\rho:\mathbb{R}\to[0;+\infty)$ on some interval $[x_0;+\infty)$, for which

$$\lim_{t\to+\infty}\rho(t)=\rho\in[0;+\infty),\ \lim_{t\to+\infty}t\rho'(t)\ln t=0$$

Example 1. *A constant function $\rho(t)\equiv\rho\in[0;+\infty)$ is a proximate* *order.*

Example 2. *A function* $\rho(t)=\rho+1/\ln(2+t)$ *is a proximate order.*

Example 3. *If the function* $\rho:\mathbb{R}\to\mathbb{R}$ *is a proximate order, then the function* $\omega(t)=t^{\rho(t)-\rho}$ *is slowly varying, because*

$$t\omega'(t)/\omega(t)=\rho(t)-\rho+t\rho'(t)\ln t.$$

Example 4. *If* $\rho\in[0;+\infty)$, *the function* $\omega:\mathbb{R}\to(0;+\infty)$ *is continuously differentiable on* $[0;+\infty)$ *and* $\lim_{t\to+\infty} t\omega'(t)/\omega(t)=0$, *then the function* $\rho(t)=\dfrac{\ln(t^{\rho}\omega(t))}{\ln t}$ *is a proximate order, because*

$$\rho(t)=\rho+\frac{\ln\omega(t)}{\ln t}\to\rho,\ t\to+\infty,$$

and

$$t\rho(t)\ln t=\frac{t\omega'(t)}{\omega(t)}+\frac{\ln\omega(t)}{\ln t}\to 0,\ t\to+\infty.$$

A proximate order $\rho:[0;+\infty)\to[0;+\infty)$ is called the proximate order of a function $\eta:[0;+\infty)\to(0;+\infty)$ if [16, 27-29, 48, 53]

$$\overline{\lim_{t\to\infty}}\frac{\eta(t)}{t^{\rho(t)}}=\sigma\in(0;+\infty).$$

For every function η that has an order $\rho\in(0;+\infty)$, there exists its proximate order $\rho:\mathbb{R}\to\mathbb{R}$ such that [16, 27-29, 48, 53]: a) $\lim_{t\to+\infty}\rho(t)=\rho$; b) $\overline{\lim_{r\to\infty}}\dfrac{\eta(r)}{r^{\rho(r)}}=\sigma\in(0;+\infty)$; c) $\dfrac{\eta(r)}{r^{\rho(r)}}\le\sigma$, $r\ge r_0$; d) $\dfrac{\eta(r_n)}{r_n^{\rho(r_n)}}=\sigma$ on some sequence (r_n), $0<r_n\uparrow+\infty$.

The proximate order $\rho:[0;+\infty)\to[0;+\infty)$ is called the formal proximate order of a function η [16, 27-29, 48, 53] if there exist numbers σ_1 and c_1 such that

$$\eta(t)\le\sigma_1 t^{\rho(t)}+c_1,\ r\ge 0. \qquad (1)$$

In this case, the number σ_1 is called the formal type of the function with respect to the proximate order $\rho:[0;+\infty)\to[0;+\infty)$, and the exact lower bound σ of all σ_1 for which there exists a constant c_1 such that (1) holds is called the type of the function η with respect to the formal proximate order $\rho:[0;+\infty)\to[0;+\infty)$, that is [16, 27-29, 48, 53]:

$$\sigma = \overline{\lim_{r\to\infty}} \frac{\eta(r)}{r^{\rho(r)}}.$$

Example 5. *A constant function* $\rho(t) \equiv \rho \in [0;+\infty)$ *is a proximate order of the function* $\eta(t) = 2t^{\rho} + t^{\rho/2}$.

Example 6. *A function* $\rho(t) = \rho + \frac{\ln\ln t}{\ln t}$ *is a proximate order of the function* $\eta(t) = 2t^{\rho}\ln t + t^{\rho}$.

1.6. Integration and differentiation of asymptotic formulas. Asymptotic equalities can generally be term-by-term integrated if natural conditions are satisfied. The justification for the possibility of term-by-term integration is based on the definitions of Landau symbols [16, 27-29, 48, 53].

***Example 1* ([16, 27-29, 48, 53]).** *If the function* $f:\mathbb{R}\to\mathbb{R}$ *is continuous on* $[0;+\infty)$, $\alpha > -1$ *and* $f(t) \sim t^{\alpha}$ *as* $t \to +\infty$, *then*

$$\int_0^x f(t)dt \sim \frac{x^{\alpha+1}}{\alpha+1}, \; x \to +\infty.$$

Indeed, $f(t) \sim t^{\alpha}$ *as* $t \to +\infty$ *if and only if* $f(t) = t^{\alpha} + o(t^{\alpha})$ *as* $t \to +\infty$ *(here, the symbol* $o(t^{\alpha})$ *denotes some function* $\eta:\mathbb{R}\to\mathbb{R}$ *such that* $\eta(t) = o(t^{\alpha})$ *as* $t \to +\infty$ *). Since*

$$(\forall\varepsilon > 0)(\exists\delta > 0)(\forall t \ge \delta): \left|o(t^{\alpha})\right| \le \varepsilon t^{\alpha},$$

we have

$$\left|\int_{\delta}^{x} o(t^{\alpha})dt\right| \le \varepsilon \frac{x^{\alpha+1}}{\alpha+1}, \; x \ge \delta.$$

Thus, from the equality

$$\int_0^x f(t)dt = \int_0^{\delta} f(t)dt + \int_{\delta}^{x} f(t)dt = O(1) + \int_{\delta}^{x} t^{\alpha}dt + \int_{\delta}^{x} o(t^{\alpha})dt =$$

$$= O(1) + \frac{x^{\alpha+1}}{\alpha+1} + \int_{\delta}^{x} o(t^{\alpha})dt, \; x \in \mathbb{R},$$

we obtain

$$\int_0^x f(t)dt - \frac{x^{\alpha+1}}{\alpha+1} = o(x^{\alpha+1}), \; x \to +\infty.$$

From this it follows the required result.

Example 2. *Let*

$$f(t)=\begin{cases}1, & t\in[0;1],\\ t^{\alpha}, & t>1, \quad \alpha<-1.\end{cases}$$

Then

$$\int_0^x f(t)dt \ge 1,\ x\ge 1,\ \frac{x^{\alpha+1}}{\alpha+1}=o(1),\ x\to+\infty.$$

Thus, the relation considered in Example 1 is not correct.

Asymptotic formulas can generally be differentiated term-by-term only when certain additional conditions are satisfied. The justification for the possibility of term-by-term differentiation is a rather complex and little-studied problem, which is related to Tauber's theorems [16, 27-29, 48, 53].

Example 3. *If* $f(t)=t+\sin t$ *then* $f(t)\sim t$ *as* $t\to+\infty$, *and* $f'(t)=1+\cos t$. *Therefore, the relation* $f'(t)\sim 1$, $t\to+\infty$, *does not hold. Thus, the asymptotic equality* $f(t)\sim t$ *as* $t\to+\infty$, *cannot, in general, be differentiated term-by-term.*

Example 4 **([16, 45, 47, 53]).** *Let* $\Delta\in\mathbb{R}$, $\rho\in(0;+\infty)$ *and the function* $f:\mathbb{R}\to\mathbb{R}$ *be represented in the form* $f(x)=\int_1^x \eta(t)d\ln t$, *where* $\eta:\mathbb{R}\to\mathbb{R}$ *is a non-decreasing and non-negative function on* $[1;+\infty)$. *Then, the conditions*

$$f(t)\sim\Delta t^{\rho},\ t\to+\infty, \tag{1}$$

and

$$\eta(t)\sim\rho\Delta t^{\rho},\ t\to+\infty, \tag{2}$$

are equivalent.

Indeed, if condition (2) holds, then

$$f(x)\sim\rho\Delta\int_1^x t^{\rho-1}dt\sim\Delta x^{\rho},\ x\to+\infty.$$

Conversely, let condition (1) hold. Then

$$\eta(r)\ln\frac{R}{r}\le f(R)-f(r)=\int_r^R\frac{\eta(t)}{t}dt\le\eta(R)\ln\frac{R}{r},\ 0<r<R<+\infty.$$

Taking $R=(1+\delta)r$ *where* $\delta>0$, *we obtain*

$$\eta(r)\le\frac{f(R)-f(r)}{\ln(R/r)}=\frac{\Delta r^{\rho}\left((1+\delta)^{\rho}-1\right)+o(r^{\rho})}{\ln(1+\delta)},\ r\to+\infty.$$

Therefore

$$\gamma := \overline{\lim_{r\to+\infty}}\eta(r)/r^{\rho} \le \frac{\Delta\left((1+\delta)^{\rho}-1\right)}{\ln(1+\delta)}.$$

Hence, $\gamma \le \rho\Delta$*. Similarly, taking* $r = R(1-\delta)$*, we get*

$$\eta(R) \ge \frac{f(R)-f(r)}{\ln(R/r)} \ge \frac{\Delta R^{\rho}\left((1-\delta)^{\rho}-1\right)+o(R^{\rho})}{-\ln(1-\delta)},\ R\to+\infty .$$

Thus

$$\gamma_1 := \underline{\lim}_{R\to+\infty}\eta(R)/R^{\rho} \ge \frac{\Delta\left((1-\delta)^{\rho}-1\right)}{-\ln(1-\delta)}.$$

Тому

$$\gamma_1 := \underline{\lim}_{R\to+\infty}\eta(R)/R^{\rho} \ge \Delta\rho .$$

Hence, $\lim_{r\to+\infty}\eta(r)/r^{\rho} = \rho\Delta$*, and we prove the required proposition.*

Example 5 ([16, 45, 47, 53]). *Let the function* $f:\mathbb{R}\to\mathbb{R}$ *be represented in the form* $f(x)=\int_1^x \eta(t)d\ln t$*, where* $\eta:\mathbb{R}\to\mathbb{R}$ *is a non-negative and non-decreasing function on* $[1;+\infty)$*. Then, the following conditions*

$$\frac{f(t)}{\ln t}\to 0,\ t\to+\infty, \tag{3}$$

and

$$\eta(t)\to 0,\ t\to+\infty, \tag{4}$$

are equivalent.

Indeed,if condition (4) is true, then

$$-\varepsilon\ln x < \int_1^x \frac{\eta(t)}{t}dt < \varepsilon\ln x$$

for every $\varepsilon>0$ *and all* $x\ge x_0(\varepsilon)$*. Therefore, condition (3) holds. Conversely, assume that condition (3) holds. Suppose that (4) does not hold. Then,* $\eta(t)\ge\varepsilon$ *for every* $\varepsilon>0$ *and all* $t\ge x_0(\varepsilon)$*. Therefore,*

$$f(x)=\int_1^x \frac{\eta(t)}{t}dt \ge O(1)+\varepsilon\ln x$$

for all sufficiently large x*, which contradicts condition (3).*

***Example 6* ([6, 16, 19]).** *Let* $\Delta\in[0;+\infty)$, $\rho\in(0;+\infty)$ *and* $f(x)=\int\limits_1^t \eta(t)d\ln t$, *where* $\eta:\mathbb{R}\to\mathbb{R}$ *is a monotonic function on* $[1;+\infty)$. *In order that for some* $\rho_2\in(0;\rho)$ *holds*

$$f(x)=\Delta x^{\rho}+o(x^{\rho_2}),\ x\to+\infty,$$

it is necessary and sufficient that for some $\rho_1\in(0;\rho)$

$$\eta(t)=\rho\Delta t^{\rho}+o(t^{\rho_1}),\ t\to+\infty.$$

Indeed, the sufficiency is established by direct verification. Now, let's prove the necessity. Since

$$\eta(r)\ln\frac{R}{r}\le f(R)-f(r)=\int\limits_r^R\frac{\eta(t)}{t}dt\le\eta(R)\ln\frac{R}{r},\ 0<r<R<+\infty,$$

then, puting $R=r+r^{\alpha}$, *where* $1+\rho_2-\rho<\alpha<1$ *and* $\max\{\rho_2-\alpha+1;\rho+\alpha-1\}<\rho_1<\rho$, *as* $r\to+\infty$ *we obtain*

$$\eta(r)\le\frac{\Delta r^{\rho}\left((1+r^{\alpha-1})^{\rho}-1\right)+o(r^{\rho_2})}{\ln(1+r^{\alpha-1})}=$$

$$=\frac{\Delta r^{\rho}\left(\rho r^{\alpha-1}+O(r^{2(\alpha-1)})\right)+o(r^{\rho_2})}{r^{\alpha-1}+O(r^{2(\alpha-1)})}=\Delta\rho r^{\rho}+o(r^{\rho_1}).$$

Therefore, $\eta(r)\le\rho\Delta r^{\rho}+o(r^{\rho_1})$ *as* $r\to+\infty$. *On the other hand, taking* $r=R-R^{\alpha}$, *as* $R\to+\infty$ *we get*

$$\eta(R)\ge\frac{\Delta R^{\rho}\left(1-(1-R^{\alpha-1})^{\rho}\right)+o(R^{\rho_2})}{-\ln(1-R^{\alpha-1})}=$$

$$=\frac{\Delta\rho R^{\rho}+O(R^{\rho-1+\alpha})+o(R^{\rho_2+1-\alpha})}{1+O(R^{\alpha-1})}=\Delta\rho R^{\rho}+o(R^{\rho_1}).$$

Thus, $\eta(R)\le\rho\Delta R^{\rho}+o(R^{\rho_1})$ *as* $R\to+\infty$, *and the necessity is proved.*

Example 7. *If* $f\in C^{(1)}[0;+\infty)$ *and* $f'(x)+f(x)\to0$ *as* $x\to+\infty$, *then* $f(x)\to0$ *and* $f'(x)\to0$ *as* $x\to+\infty$. *Indeed, let* $f'+f=\varepsilon$. *Then* $\varepsilon(x)\to0$ *as* $x\to+\infty$, *and the function* f *is a solution of the equation* $f'+f=\varepsilon$. *Solving the last equation by the method of variation of arbitrary constants, we conclude that*

$$f(x)=ce^{-x}+e^{-x}\int_0^x \varepsilon(t)e^t dt .$$

From here, by using, for example, L'Hôpital's rule, we obtain $f(x)\to 0$ *as* $x\to+\infty$. *Thus,* $f'(x)\to 0$ *as* $x\to+\infty$.

1.7. Estimation of zeros of functions and roots of equations. At studying many problems, it is necessary to be able to investigate the existence of zeros of a function, their multiplicity, and find them with sufficient accuracy. A zero of a function $f:\mathbb{R}\to\mathbb{R}$ is called [3, 26] a number $a\in\mathbb{R}$ such that $f(a)=0$, i.e., a zero of the function $f:\mathbb{R}\to\mathbb{R}$ is a root of the equation $f(x)=0$. A function $f:\mathbb{R}\to\mathbb{R}$ that is m times continuously differentiable at a point $a\in\mathbb{R}$ is said to have a zero of order $m\in\mathbb{N}$ or a zero of multiplicity $m\in\mathbb{N}$ if [3, 26]

$$f(a)=f'(a)=...=f^{(m-1)}(a)=0,\ f^{(m)}(a)\neq 0. \qquad (1)$$

A zero of order $m=1$ is called a simple zero.

Theorem 1 ([3, 26]). *Let* f *be a polynomial of degree* n *and* $m\le n$. *Then, the following conditions are equivalent: 1) at the point* $a\in\mathbb{R}$ *the polynomial* f *has a zero of order* m; *2)* $f(x)=(x-a)^m g(x)$, *where* g *is a polynomial of degree* $n-m$ *and* $g(a)\neq 0$; *3)* $f(x)=\sum_{k=m}^{n} b_k (x-a)^k$, *where* $b_m\neq 0$.

Example 1. *A polynomial* $f(x)=(x+2)^4(x-5)$ *of degree* $n=5$ *has a zero of order* $m=4$ *at a point* $a_1=-2$ *ana zero of order* $m=1$ *at a point* $a_2=5$.

Remark 1. *According to the fundamental theorem of algebra, every polynomial of degree* n *has at most* n *real zeros (and exactly* n *complex zeros, counting each zero according to its multiplicity).*

If a function $f:[a;b]\to\mathbb{R}$ is continuous on a interval $[a;b]$ and takes values of opposite signs at the endpoints of the interval, then by the Bolzano-Cauchy theorem [4], the equation $f(x)=0$ has at least one root on $[a;b]$. The question is how to find this root with sufficient accuracy. There are several methods for this. Before choosing a specific approximation method, it is important to determine how many roots the equation has, find the multiplicity of each root, and identify the intervals that contain exactly one root [4, 16].

Example 2. *The function* $f(x)=x^3+x+1$ *is continuous on* $\mathbb{R}$, $\lim_{x\to+\infty} f(x)=+\infty$, $\lim_{x\to-\infty} f(x)=-\infty$ *and* $f'(x)=3x^2+1$. *Therefore, this* *function is increasing on* $\mathbb{R}$. *Hence, the equation* $x^3+x+1=0$ *has a unique real root. In addition,* $f''(x)=6x$ *and* $f(0)=1\neq 0$. *Thus, the root is simple. Since* $f(-1)=-1<0$ *and* $f(0)=1>0$, *then the root lies within the interval* $(-1;0)$.

Example 3. *Let's find the values of* $a\in\mathbb{R}$ *for which the equation* $x^3-3ax^2+3a^2x-1=0$ *has real roots of multiplicity* $m\geq 2$ *Let* $f(x)=x^3-3ax^2+3a^2x-1$. *Then* $f'(x)=3x^2-6ax+3a^2$. *If such a* $a\in\mathbb{R}$ *exists, then the system* $\begin{cases} f(x)=0, \\ f'(x)=0, \end{cases}$ *is consistent, meaning the system*

$$\begin{cases} x^3-3ax^2+3a^2x-1=0, \\ 3x^2-6ax+3a^2=0. \end{cases}$$

From the second equation, we obtain $x=a$ *and, therefore,* $a^3-3aa^2+3a^2a-1=0$. *Hence,* $a=1$ *and* $f(x)=(x-1)^3$. *Thus, we conclude that the equation has real roots only in the case* $a=1$ *and in this case, the equation has exactly one real root* $x=1$ *of multiplicity* $m=3$.

Example 4. *Let* $f(x)=x^3-3ax+1$. *Then, this function is a continuous on* $\mathbb{R}$, $\lim_{x\to+\infty} f(x)=+\infty$, $\lim_{x\to-\infty} f(x)=-\infty$ *and* $f'(x)=3x^2-3a$. *Thus, the equation* $x^3-3ax+1=0$ *has at least one real root. If* $a\leq 0$, *then the function is increasing, and therefore it has exactly one real root.*

1.8. Asymptotics of inverse functions and roots of equations. To find the formula $y=f^{-1}(x)$ that defines the function $f^{-1}:\mathbb{R}\to\mathbb{R}$ inverse to the function $f:\mathbb{R}\to\mathbb{R}$, one needs to solve the equation $y=f(x)$ to $x=f^{-1}(y)$ and then interchange x and y. As a result, the desired formula is obtained. This process involves using known identities such as $f^{-1}(f(x))=x$, $x\in D(f)$ and $f(f^{-1}(y))=y$, $y\in E(f)$. However, solving the equation $y=f(x)$ is often challenging. For this reason, in many cases, it is more practical to find an asymptotic formula [4, 16, 27-29, 48, 53]. A similar approach is used when solving equations $f(x)=a$. If such an equation has an infinite number of roots, it is often possible to represent the set $\mathbb{R}$ as a union of a countable number of pairwise disjoint intervals, within each of which the function f is invertible. Then, one can determine the asymptotics of the roots belonging to the considered intervals.

Example 1. *If* $f(x)=4x-8$, *then we have the equation* $y=4x-8$, *from which it follows that* $x = \frac{1}{4}y + 2$. *Therefore,* $f^{-1}(x) = \frac{1}{4}x + 2$ *and* $y = \frac{1}{4}x + 2$ *the desired formula.*

Example 2. *The function* $f(x) = 3x + \sqrt{x}$ *is increasing and continuous on the interval* $(0;+\infty)$. *Therefore, it has an inverse function* f^{-1} *and* $f^{-1}(y) \to +\infty$ *as* $y \to +\infty$. *In addition, we have* $f(f^{-1}(y)) = 3f^{-1}(y) + \sqrt{f^{-1}(y)}$. *Thus,* $y = 3(1+o(1))f^{-1}(y)$ *as* $y \to +\infty$, *and* $f^{-1}(x) = (1+o(1))x/3$ *as* $x \to +\infty$.

Example 3. *If* $f:\mathbb{R}\to\mathbb{R}$ *and* $\varphi:\mathbb{R}\to\mathbb{R}$ *are two invertible functions,* $f(x) = (1+o(1))\varphi(x)$ *as* $x \to 0$ *and* $\varphi^{-1}(y) = o(1)$ *as* $y \to 0$, *then* $f^{-1}(y) = \varphi^{-1}((1+o(1))y)$ *as* $y \to 0$.

Indeed, $x = \varphi^{-1}(y) \to 0$ *as* $y \to 0$. *Therefore, we have* $f(\varphi^{-1}(y))) = (1+o(1))\varphi(\varphi^{-1}(y))$ *and* $f(\varphi^{-1}(y))) = (1+o(1))y$ *as* $y \to 0$. *In particular, if* $f(x) = (1+o(1))x$ *as* $x \to 0$, *then* $f^{-1}(x) = (1+o(1))x$ *as* $x \to 0$.

Example 4. *If the function* $f:(0;1)\to\mathbb{R}$ *is invertible and*

$$f(x) = ax + bx^2 + o(x^2),\ x \to 0,$$

then $f(x) = (1+o(1))ax$, *if* $a \neq 0$ *and* $x \to +0$. *Therefore* $f^{-1}(x) = (1+o(1))x/a$ *as* $x \to +0$. *If* $a = 0$ *and* $b \neq 0$, *then* $f(x) = (1+o(1))bx^2$ *and* $f^{-1}(x) = (1+o(1))\sqrt{x/b}$ *as* $x \to +0$.

Example 5 ([16]). *The function* $f(x) = x + \ln x$ *is increasing and continuous on* $(0;+\infty)$. *Therefore, it has an inverse function. In this case,* $f(x) \to +\infty$ *as* $x \to +\infty$, *and* $f(x) \to -\infty$ *as* $x \to 0+$. *To find the inverse function, we consider the equation* $y = x + \ln x$. *Solving this equation explicitly might be challenging or even impossible. However,*

$$f^{-1}(x) = (1+o(1))x,\ x \to +\infty,$$

$$f^{-1}(x) = x - \ln x + o(1),\ x \to +\infty,$$

$$f^{-1}(x) = x - \ln x + \frac{\ln x}{x} + o\left(\frac{\ln x}{x}\right),\ x \to +\infty,$$

We derive these asymptotic formulas as follows:

$$y = x + \ln x,\ y = (1+o(1))x,\ x = (1+o(1))y,\ y \to +\infty;$$

$$x = y - \ln x = y - \ln(1 + o(1))y = y - \ln y + o(1)\,,\ y \to +\infty\,;$$
$$x = y - \ln x = y - \ln\left(y - \ln y + o(1)\right) = y - \ln\left(y - \ln y\right) + \ln\left(1 + \frac{o(1)}{y - \ln y}\right) =$$
$$= y - \ln y - \ln\left(1 - \frac{\ln y}{y}\right) + \ln\left(1 + \frac{o(1)}{y - \ln y}\right) =$$
$$= y - \ln y + \frac{\ln y}{y} + o\left(\frac{\ln y}{y}\right),\ y \to +\infty\,;$$
$$\ldots.$$

The asymptotic formulas
$$f^{-1}(x) = e^{x + o(1)},\ x \to -\infty\,,$$
$$f^{-1}(x) = e^{x} + O\left(e^{2x}\right),\ x \to -\infty\,,$$
are obtained as follows:
$$y = x + \ln x,\ y = \ln x + o(1),\ x = e^{y + o(1)},\ y \to -\infty\,;$$
$$x = e^{y - x} = e^{y - e^{y + o(1)}} = e^{y} e^{-e^{y + o(1)}} = e^{y}\left(1 + O\left(e^{y}\right)\right) = e^{y} + O\left(e^{2y}\right),\ y \to -\infty\,.$$
Hence, if $x = x(y)$ *is a solution of the equation* $x + \ln x = y$, *then* $x = y - \ln y + o(1)$ *as* $y \to +\infty$, *and* $x = e^{y} + O\left(e^{2y}\right)$ *as* $y \to -\infty$.

Example 6 ([16]). *Let* x_n *be the root of the equation* $\operatorname{tg} x = 1/x$, *which lies in the interval* $(-\pi/2 + \pi n; \pi/2 + \pi n)$. *From geometric considerations, it follows that for* $n \ge 2$ *on this interval the considered equation has a unique root* $x_n = \pi n + \Delta_n$, $\Delta_n \in (-\pi/2; \pi/2)$. *Besides,* $\operatorname{tg}\Delta_n = \dfrac{1}{x_n}$, $\Delta_n = arctg\dfrac{1}{x_n}$ *and* $x_n = \pi n + \operatorname{arctg}\dfrac{1}{x_n}$. *Hence,* $x_n = \pi n + o(1)$ *as* $n \to +\infty$. *But* $\operatorname{arctg} x = x + o\left(x^2\right)$ *as* $x \to 0$. *Thus,*
$$x_n = \pi n + \frac{1}{\pi n + o(1)} + o\left(\frac{1}{n^2}\right) = \pi n + \frac{1}{\pi n}\frac{1}{1 + o\left(\frac{1}{n}\right)} + o\left(\frac{1}{n^2}\right) =$$
$$= \pi n + \frac{1}{\pi n} + o\left(\frac{1}{n^2}\right),\ n \to +\infty\,.$$

1.9. Asymptotics of integrals with a variable upper limit. To find the asymptotics of the integral [5, 16, 27-29, 48, 53]

$$\int_a^x f(t)dt = (1+o(1))\eta(x)\,,\ x\to a\,,$$

the search for the function η can be carried out using the trial and error method with the application of L'Hopital's rules or other techniques.

Example 1. *The asymptotic equality*

$$\int_1^x \frac{e^t}{t}dt = (1+o(1))\frac{e^x}{x} \ \text{as}\ x\to+\infty\,,$$

is fulfilled, because

$$\lim_{x\to+\infty}\frac{\int_1^x \frac{e^t}{t}dt}{\frac{e^x}{x}} = \lim_{x\to+\infty}\frac{\left(\int_1^x \frac{e^t}{t}dt\right)'}{\left(\frac{e^x}{x}\right)'} = \lim_{x\to+\infty}\frac{\frac{e^x}{x}}{\frac{e^x x - e^x}{x^2}} = 1\,.$$

Example 2. *For any* $\alpha\in\mathbb{R}$, $\mu>-1$ *and* $\varepsilon>0$, *the following holds:*

$$\int_{\varepsilon x}^{+\infty} e^{-u}u^{\mu}du = o\left(\frac{1}{x^{\alpha}}\right) \ \text{as}\ x\to+\infty\,,$$

because

$$\lim_{x\to+\infty}\frac{\int_{\varepsilon x}^{+\infty} e^{-u}u^{\mu}du}{\frac{1}{x^{\alpha}}} = \lim_{x\to+\infty}\frac{\left(\int_{\varepsilon x}^{+\infty} e^{-u}u^{\mu}du\right)'}{\left(\frac{1}{x^{\alpha}}\right)'} = \lim_{x\to+\infty}\frac{-\varepsilon e^{-\varepsilon x}(\varepsilon x)^{\mu}}{-\alpha x^{-\alpha-1}} = 0\,.$$

Example 3 **([16]).** *If the function* $h:\mathbb{R}\to\mathbb{R}$ *is twice continuously differentiable on* $[0;+\infty)$, $h(x)\to+\infty$ *and* $h''(x)=o\left(h'^2(x)\right)$ *as* $x\to+\infty$, *then*

$$\int_x^{+\infty} e^{-h(t)}dt = (1+o(1))\frac{e^{-h(x)}}{h'(x)}\,,\ x\to+\infty\,.$$

In fact,

$$-\frac{h(x)}{2}-c_1\le \ln h'(x)\le \frac{h(x)}{2}+c_1\,,\ c_1>0\,.$$

Therefore,

$$\int_0^{+\infty} e^{-h(t)}dt = \int_0^{+\infty}\frac{e^{-h(t)}dh(t)}{h'(t)} < +\infty$$

and

$$\frac{e^{-h(x)}}{h'(x)} \to 0\,,\ x \to +\infty\,.$$

Thus

$$\lim_{x\to+\infty} \frac{\int\limits_x^{+\infty} e^{-h(t)}dt}{\dfrac{e^{-h(x)}}{h'(x)}} = \lim_{x\to+\infty} \frac{h'^2(x)}{h'^2(x)+h''(x)} = 1\,.$$

1.10. Asymptotics of sums. The asymptotics of sums

$$\sum_{k=1}^{n} a(k)\,,\ \sum_{k=n+1}^{\infty} a(k)\,,$$

can be found by comparing them with integrals

$$\int\limits_1^n a(t)dt\,,\ \int\limits_{n+1}^{+\infty} a(t)dt\,,$$

expressing the sum in the form of a Stieltjes integral, using the Stolz theorem, and applying other methods [5, 16, 27-29, 48, 53].

***Example 1* ([16]).** *Let a function* $a:(0;+\infty)\to[0;+\infty)$ *be non-decresing on* $(0;+\infty)$. *Then*

$$a(k) \le \int\limits_k^{k+1} a(t)dt \le a(k+1)\,.$$

Therefore,

$$\sum_{k=1}^{n} a(k) = a(n) + \sum_{k=1}^{n-1} a(k) \le a(n) + \sum_{k=1}^{n-1}\int\limits_k^{k+1} a(t)dt = a(n) + \int\limits_1^n a(t)dt\ ,$$

$$\sum_{k=1}^{n} a(k) = a(1) + \sum_{k=1}^{n-1} a(k+1) \ge a(1) + \sum_{k=1}^{n-1}\int\limits_k^{k+1} a(t)dt = a(1) + \int\limits_1^n a(t)dt\,.$$

Thus,

$$\sum_{k=1}^{n} a(k) = \int\limits_1^n a(t)dt + O(1) + O(a(n))\,,\ n\to\infty\,.$$

If the function $a:(0;+\infty)\to[0;+\infty)$ *is non-increasing on* $(0;+\infty)$, *then*

$$a(k+1) \le \int\limits_k^{k+1} a(t)dt \le a(k)\,.$$

Therefore,

$$\sum_{k=1}^{n} a(k) = a(1) + \sum_{k=1}^{n-1} a(k+1) \le a(1) + \sum_{k=1}^{n-1} \int_{k}^{k+1} a(t)dt = a(1) + \int_{1}^{n} a(t)dt ,$$

$$\sum_{k=1}^{n} a(k) = a(n) + \sum_{k=1}^{n-1} a(k) \ge a(n) + \sum_{k=1}^{n-1} \int_{k}^{k+1} a(t)dt = a(n) + \int_{1}^{n} a(t)dt \ .$$

Hence, if the function $a:(0;+\infty) \to [0;+\infty)$ *is monotonic, then*

$$\sum_{k=1}^{n} a(k) = \int_{1}^{n} a(t)dt + O(1) + O(a(n)) \, , \ n \to \infty \, .$$

Example 2 **([16]).** *Since*

$$\sum_{k=1}^{n} \frac{1}{k} < 1 + \int_{1}^{n} \frac{1}{t} dt = 1 + \ln n \, , \ \sum_{k=1}^{n} \frac{1}{k} > \frac{1}{n} + \int_{1}^{n} \frac{1}{t} dt = \frac{1}{n} + \ln n \, ,$$

we have

$$\sum_{k=1}^{n} \frac{1}{k} = \int_{1}^{n} \frac{1}{t} dt + O(1) = \ln n + O(1) \, , \ n \to \infty \, .$$

More precise estimates can be obtained. In particular, there exists a limit:

$$\gamma_0 := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right) = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \ln\left(1 + \frac{1}{k} \right) \right). \qquad (1)$$

Further,

$$\sum_{k=1}^{n} \frac{1}{k} - \ln n - \gamma_0 = \frac{\theta_n}{n} \, , \ \theta < \theta_n < 1 \, , \ n > 1 \, . \qquad (2)$$

Indeed,

$$\sum_{k=1}^{n} \frac{1}{k} - \ln n = \sum_{k=1}^{n} \left(\frac{1}{k} - \ln\left(1 + \frac{1}{k} \right) \right) + \sum_{k=1}^{n} \ln \frac{k+1}{k} - \ln n =$$

$$= \sum_{k=1}^{n} \left(\frac{1}{k} - \ln\left(1 + \frac{1}{k} \right) \right) + \ln \frac{n+1}{n} \, .$$

From this it follows (1). In addition,

$$s_n := \sum_{k=1}^{n} \frac{1}{k} - \ln n - \gamma_0 = \ln\left(1 + \frac{1}{n} \right) - \sum_{k=n+1}^{\infty} \left(\frac{1}{k} - \ln\left(1 + \frac{1}{k} \right) \right).$$

Since

$$\frac{1}{2k(2k+1)} \le \frac{1}{2k^2} - \frac{1}{3k^3} \le \frac{1}{k} - \ln\left(1 + \frac{1}{k} \right) \le \frac{1}{2k^2} \le \frac{1}{2k(2k-1)} \, ,$$

$$\frac{1}{2n+1} \le \sum_{k=n+1}^{\infty} \left(\frac{1}{k} - \ln\left(1 + \frac{1}{k} \right) \right) \le \frac{1}{2n-1} \, ,$$

$$\frac{1}{n}-\frac{1}{2n(2n-1)}<\ln\left(1+\frac{1}{n}\right)<\frac{1}{n}-\frac{1}{2n(2n+1)},$$

we have

$$\frac{1}{n}-\left(\frac{1}{2n(2n-1)}-\frac{1}{2n+1}\right)<s_n<\frac{1}{n}-\left(\frac{1}{2n(2n+1)}-\frac{1}{2n-1}\right),$$

from which we obtain (2). The constant $\gamma_0=\ln\sqrt{2\pi}=0{,}57...$ *is called Euler's constant.*

Example 3 ([16]). *Let the function* $a:(0;+\infty)\to[0;+\infty)$ *be non-increasing on* $(0;+\infty)$. *Then*

$$a(k+1)\le\int_k^{k+1}a(t)dt\le a(k).$$

Consequently,

$$\sum_{k=n+1}^{\infty}a(k)=\sum_{k=n}^{\infty}a(k+1)\le\sum_{k=n}^{\infty}\int_k^{k+1}a(t)dt=\int_n^{+\infty}a(t)dt,$$

$$\sum_{k=n+1}^{\infty}a(k)=-a(n)+\sum_{k=n}^{\infty}a(k)\ge-a(n)+\sum_{k=n}^{\infty}\int_k^{k+1}a(t)dt=-a(n)+\int_n^{+\infty}a(t)dt.$$

Thus,

$$\sum_{k=n+1}^{\infty}a(k)=\int_n^{+\infty}a(t)dt+O(a(n)),\ n\to\infty.$$

In particular,

$$\sum_{k=n+1}^{\infty}\frac{1}{k^3}=\int_n^{+\infty}\frac{1}{t^3}dt+O(1/n^3)=\frac{1}{2n^2}+O(1/n^3),\ n\to\infty,$$

$$\sum_{k=1}^{n}e^{\sqrt{k}}=\int_1^{n}e^{\sqrt{t}}dt+O(e^{\sqrt{n}})=2(1+o(1))\sqrt{n}e^{\sqrt{n}},\ n\to\infty.$$

Example 4 ([16]). *Let* $x_n=\sum_{k=0}^{n-1}k^{\alpha}q^k$, $y_n=n^{\alpha}q^n$ *with* $q>1$ *and* $\alpha\in\mathbb{R}$. *Then* $y_n\to+\infty$, $x_n-x_{n-1}=(n-1)^{\alpha}q^{n-1}$,

$$y_n-y_{n-1}=n^{\alpha}q^n-(n-1)^{\alpha}q^{n-1}=$$

$$=n^{\alpha}q^n\left(1-\left(1-\frac{1}{n}\right)^{\alpha}/q\right)=n^{\alpha}q^{n-1}(q-1)(1+o(1))\to+\infty.$$

as $n\to\infty$. *Further,* $\dfrac{x_n-x_{n-1}}{y_n-y_{n-1}}=\dfrac{1}{q-1}(1+o(1))$ *as* $n\to\infty$, *and by Stolz's theorem, we get* $\dfrac{x_n}{y_n}=\dfrac{1}{q-1}(1+o(1))$ *as* $n\to\infty$. *Furthermore,*

$$\sum_{k=0}^{n-1}k^\alpha q^k=\frac{n^\alpha q^n}{q-1}(1+o(1)),\ n\to\infty.$$

Example 5 ([16]). *We prove that* $\sum_{k=n}^{\infty}e^{-\sqrt{k}}=2(1+o(1))\sqrt{n}e^{-\sqrt{n}}$ *as* $n\to\infty$. *Indeed, let* $x_n=\sum_{k=n}^{\infty}e^{-\sqrt{k}}$ *and* $y_n=\sqrt{n}e^{-\sqrt{n}}$. *Then*

$$x_n-x_{n-1}=-e^{-\sqrt{n-1}},\ y_n-y_{n-1}=\sqrt{n}e^{-\sqrt{n}}-\sqrt{n-1}e^{-\sqrt{n-1}},$$

$$\frac{x_n-x_{n-1}}{y_n-y_{n-1}}=\frac{-e^{-\sqrt{n-1}}}{\sqrt{n}e^{-\sqrt{n}}-\sqrt{n-1}e^{-\sqrt{n-1}}}=\frac{-1}{\sqrt{n}\left(e^{\sqrt{n-1}-\sqrt{n}}-\sqrt{\frac{n-1}{n}}\right)}=$$

$$=\frac{-1}{\sqrt{n}\left(e^{-\frac{1}{\sqrt{n-1}+\sqrt{n}}}-\sqrt{1-\frac{1}{n}}\right)}=$$

$$=\frac{-1}{\sqrt{n}\left(1-\frac{1}{\sqrt{n-1}+\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right)-1+\frac{1}{2n}+o\left(\frac{1}{n}\right)\right)}=2(1+o(1)),$$

as $n\to\infty$. *Hence, according to the Stolz theorem, we obtain*

$$\frac{x_n}{y_n}=2(1+o(1)),\ n\to\infty.$$

1.11. Asymptotic estimates of some parameter-dependent integrals. Asymptotic estimates of many integrals

$$I(x)=\int_a^b\eta(t)e^{-xS(t)}dt$$

and others can be found through integration by parts. In more complex cases, it is advisable to use Laplace's method [5, 16, 27-29, 48, 53]. The main idea of Laplace's method is that for a given function S, which has one minimum point t_0 on an interval $[a;b]$, for large positive values x, the integrand has a large maximum at this point t_0, and therefore, this integral differs little from its integral over a small neighborhood Δ_x of the point t_0. The latter integral, in turn, differs little from

$$\eta(t_0)e^{-xS(t_0)}\int_{\Delta_x} dt\,,$$

more precise, from the contribution of the point t_0 itself. The justification of these two moments constitutes the essence of Laplace's method. With a suitable substitution in the integration variable, similar methods can be used to find the asymptotics of other types of integrals. When applying this method, the Gamma function [5, 16, 27-29, 48]

$$\Gamma(x)=\int_0^{+\infty} e^{-t}t^{x-1}dt$$

and some of its properties such as $\Gamma(1+x)=x\Gamma(x)$, $\Gamma(1+n)=n!$, $\Gamma(1/2)=\int_0^{+\infty} e^{-t}t^{-1/2}dt=2\int_0^{+\infty} e^{-u^2}du=\sqrt{\pi}$, are often used.

Example 1. *For any* $\mu>-1$ *and* $x>0$, *we have*

$$\int_0^{+\infty} e^{-tx}t^{\mu}dt=\frac{\Gamma(\mu+1)}{x^{\mu+1}}\,.$$

In particular,

$$\int_0^{+\infty} e^{-tx}dt=\frac{1}{x},\ \ x>0\,.$$

Example 2. *For any* $\delta>0$, $\mu>-1$ *and* $\alpha>0$, *the following holds:*

$$\int_\delta^{+\infty} e^{-tx}t^{\mu}dt=o(1/x^{\alpha})\ \text{ as }\ x\to+\infty\,.$$

Indeed,

$$\int_\delta^{+\infty} e^{-tx}t^{\mu}dt=\int_\delta^{+\infty} e^{-t}e^{-t(x-1)}t^{\mu}dt\le e^{-\delta(x-1)}\int_\delta^{+\infty} e^{-t}t^{\mu}dt\,,$$

if $x>1$ *and* $e^{-\delta x}=o(1/x^{\alpha})$ *as* $x\to+\infty$.

Example 3. *For any* $\varepsilon>0$ *holds*

$$\int_0^{\varepsilon} e^{-tx}dt=\int_0^{+\infty} e^{-tx}dt-\int_\varepsilon^{+\infty} e^{-tx}dt=\frac{1}{x}-\frac{1}{x}e^{-\varepsilon x}=\frac{1}{x}+o\left(\frac{1}{x}\right)\ \text{ as }\ x\to+\infty\,,$$

Example 4. *For any* $\alpha\in\mathbb{R}$ *and* $\varepsilon>0$, *we have*

$$\int_0^{\varepsilon} e^{-tx}tdt = -\frac{t}{x}e^{-tx}\Big|_0^{\varepsilon} + \frac{1}{x}\int_0^{\varepsilon} e^{-tx}dt = \frac{1}{x}\left(-\varepsilon e^{-\varepsilon x} + \int_0^{\varepsilon} e^{-tx}dt\right) =$$

$$= \frac{1}{x}\left(\frac{1}{x} + o\left(\frac{1}{x}\right)\right) = \frac{1}{x^2} + o\left(\frac{1}{x^2}\right) \ as \ x \to +\infty .$$

Example 5. *For any* $\alpha \in \mathbb{R}$, $\mu > -1$ *and* $\varepsilon > 0$, *the following holds:*

$$\int_0^{\varepsilon} e^{-tx}t^{\mu}dt = \int_0^{+\infty} e^{-tx}t^{\mu}dt - \int_{\varepsilon}^{+\infty} e^{-tx}t^{\mu}dt = \frac{1}{x^{\mu+1}}\int_0^{+\infty} e^{-u}u^{\mu}du - \int_{\varepsilon}^{+\infty} e^{-tx}t^{\mu}dt =$$

$$= \frac{\Gamma(\mu+1)}{x^{\mu+1}} - \int_{\varepsilon}^{+\infty} e^{-tx}t^{\mu}dt = \frac{\Gamma(\mu+1)}{x^{\mu+1}} - \frac{1}{x^{\mu+1}}\int_{\varepsilon x}^{+\infty} e^{-u}u^{\mu}du =$$

$$= \frac{\Gamma(\mu+1)}{x^{\mu+1}} + o\left(\frac{1}{x^{\alpha}}\right) \ as \ x \to +\infty .$$

Example 6 **([16, 27-29, 48]).** *Let the function* $\eta : [0;+\infty) \to \mathbb{R}$ *satisfy the following conditions: 1) it is continuous on the interval* $[0;+\infty)$ *; 2) there exists a constant* c_1 *such that* $|\eta(t)| \le c_1 e^{c_1 t}$ *for* $t \in [0;+\infty)$ *; 3) the function* η *can be expanded into uniformly convergent series* $\eta(t) = \sum_{k=0}^{\infty} d_k t^k$ *on some interval* $[0;\varepsilon]$, *where* $\varepsilon > 0$. *Then, for any* $\alpha > -1$ *and* $n \in \mathbb{Z}_+$ *holds*

$$\int_0^{+\infty} e^{-tx}t^{\alpha}\eta(t)dt = \sum_{k=0}^{n} d_k \frac{\Gamma(k+\alpha+1)}{x^{k+\alpha+1}} + O\left(\frac{1}{x^{n+\alpha+1}}\right) \ as \ x \to +\infty .$$

Indeed,

$$\int_0^{+\infty} e^{-tx}t^{\alpha}\eta(t)dt = \int_0^{\varepsilon} e^{-tx}t^{\alpha}\eta(t)dt + \int_{\varepsilon}^{+\infty} e^{-tx}t^{\alpha}\eta(t)dt .$$

In this case,

$$\left|\int_{\varepsilon}^{+\infty} e^{-tx}t^{\alpha}\eta(t)dt\right| \le c_1 \int_{\varepsilon}^{+\infty} e^{-t(x-2c_1)}t^{\alpha}e^{-tc_1}dt \le c_1 e^{-\varepsilon(x-2c_1)} \int_{\varepsilon}^{+\infty} t^{\alpha}e^{-tc_1}dt .$$

Therefore,

$$\int_{\varepsilon}^{+\infty} e^{-tx}t^{\alpha}\eta(t)dt = O\left(\frac{1}{x^{n+\alpha+1}}\right), \ x \to +\infty .$$

Further,

$$\int_0^{\varepsilon} e^{-tx}t^{\alpha}\sum_{k=0}^{n} d_k t^k dt = \sum_{k=0}^{n} d_k \int_0^{\varepsilon} e^{-tx}t^{\alpha+k}dt = \sum_{k=0}^{n} d_k \int_0^{+\infty} e^{-tx}t^{\alpha+k}dt + O\left(\frac{1}{x^{n+\alpha+1}}\right) =$$

$$= \sum_{k=0}^{n} d_k \frac{\Gamma(k+\alpha+1)}{x^{k+\alpha+1}} + O\left(\frac{1}{x^{n+\alpha+1}}\right),\ x \to +\infty .$$

Furthermore,

$$t^{\alpha}\eta(t) - t^{\alpha}\sum_{k=0}^{n} d_k t^k = O(t^{n+1+\alpha}),\ t \in [0;\varepsilon].$$

Hence,

$$\left|\int_0^{\varepsilon} e^{-tx}t^{\alpha}\eta(t)dt - \int_0^{\varepsilon} e^{-tx}t^{\alpha}\sum_{k=0}^{n} d_k t^k dt\right| \le c_2\int_0^{\varepsilon} e^{-tx}t^{n+1+\alpha}dt = \frac{c_2}{x^{n+2+\alpha}}\int_0^{\varepsilon x} e^{-u}u^{n+1+\alpha}du .$$

Example 7. *For* $x > 0$*, we have*

$$\int_{-\infty}^{+\infty} e^{-t^2 x}dt = \frac{1}{\sqrt{x}}\int_{-\infty}^{+\infty} e^{-u^2}du = \sqrt{\frac{\pi}{x}} .$$

Example 8. *For any* $\varepsilon \in \mathbb{R}$ *holds*

$$\int_{-\varepsilon}^{\varepsilon} e^{-t^2 x}dt = \sqrt{\frac{\pi}{x}} + o\left(\frac{1}{\sqrt{x}}\right)$$

as $x \to +\infty$*, because*

$$\int_{-\varepsilon}^{\varepsilon} e^{-t^2 x}dt = 2\int_0^{\varepsilon} e^{-t^2 x}dt = \int_0^{\varepsilon^2} e^{-ux}u^{-1/2}du = \int_0^{+\infty} u^{-1/2}e^{-ux}du - \int_{\varepsilon^2}^{+\infty} u^{-1/2}e^{-ux}du =$$

$$= \frac{\Gamma(1/2)}{\sqrt{x}} - \frac{1}{\sqrt{x}}\int_{x\varepsilon^2}^{+\infty} v^{-1/2}e^{-v}dv .$$

1.12. Self-control questions.

1. Explain the meaning of the symbols: a) “ $f(x) = o(1)$, $x \to a$ ”; b) “ $f(x) = o(\varphi(x))$, $x \to a$ ”; c) “ $f(x) = O(1)$, $x \to a$ ”; d) “ $f(x) = O(\varphi(x))$, $x \to a$ ”; e) “ $f(x) \asymp \varphi(x)$, $x \to a$ ”; f) “ $f(x) = O(1)$, $x \in E$ ”; g) “ $f(x) = O(\varphi(x))$, $x \in E$ ”; h)“ $f(x) \sim \varphi(x)$, $x \to a$ ”.
2. Formulate the definition of the order of a function.
3. Formulate the definition of the type of a function.
4. Formulate the definition of a slowly varying function.
5. Formulate the definition of a slowly increasing function.
6. Formulate the definition of a proximate order.
7. What are the main methods used for finding the asymptotics of integrals with a variable upper bound and the asymptotics of sums?

8. Formulate the Stolz theorem.
9. What is the essence of Laplace's method?

1.13. Exercises and problems.

1. Determine whether the following statements are true:

1. $\sin(x+o(x^2))=x+o(x^2)$, $x\to 0$.
2. $\cos(x+o(x^2))=1-\dfrac{1}{2}x^2+o(x^2)$, $x\to 0$.
3. $\sin(x+o(x^2))=x+o(x^2)$, $x\to 0$.
4. $\mathrm{tg}(x+o(x^2))=x+o(x^2)$, $x\to 0$.
5. $(1+x)^5=O(x^5)$, $x\to\infty$.
6. $\ln x=o(x^{\alpha})$, $x\to+\infty$, $\alpha\le 0$.
7. $\ln x=o(x^2)$, $x\to+\infty$.
8. $\ln x=o(x^2)$, $x\to 0+$.
9. $xo(x)=o(x^2)$, $x\to 0$.
10. $x^3 o(1/x)=o(x^2)$, $x\to 0$.
11. $x^2+o(x)=O(x^2)$, $x\to\infty$.
12. $o(O(x))=o(x)$, $x\to 0$.
13. $o(x^2)+O(x^3)=o(x^2)$, $x\to 0$.

2. Find a number c_0 such that $f(x)=c_0(x-a)^m+o((x-a)^m)$ as $x\to a$ and $m\in\mathbb{Z}_+$:

1. $f(x)=\sin(\cos x)$, $a=0$.
2. $f(x)=\cos(\sin x)$, $a=0$.
3. $f(x)=\sin(\mathrm{tg}^2 x)$, $a=0$.
4. $f(x)=\mathrm{tg}^2(\sin x)$, $a=0$.
5. $f(x)=\sin(\ln\cos x)$, $a=0$.
6. $f(x)=\cos(\arcsin^2 x)$, $a=0$.
7. $f(x)=\arcsin^2(\mathrm{tg}\, x)$, $a=0$.
8. $f(x)=\mathrm{tg}^2(\arccos x)$, $a=0$.
9. $f(x)=\arcsin^2(x+x^2+o(x^2))$, $a=0$.
10. $f(x)=\mathrm{tg}^2(x+O(x^3))$, $a=0$.

3. Determine whether the following asymptotic formulas $f(2x)\sim f(x)$, $f(x)=o(f(2x))$, $f(x+1)\sim f(x)$ and $f(x)=o(f(x+1))$ are valid as $x\to+\infty$:

1. $f(x)=\sqrt{x}$.
2. $f(x)=x^2$.
3. $f(x)=e^{\sqrt{x}}$.
4. $f(x)=e^{2x}$.
5. $f(x)=\ln x$.
6. $f(x)=\ln\ln x$.
7. $f(x)=\ln^2 x$.
8. $f(x)=\sqrt{\ln x}$.
9. $f(x)=\sqrt[3]{x}$.
10. $f(x)=\ln^p x$, $p\in\mathbb{R}$.
11. $f(x)=\ln^p \ln^q x$, $p,q\in\mathbb{R}$.
12. $f(x)=e^{p\sqrt{\ln x}}$, $p\in\mathbb{R}$.
13. $f(x)=e^{\ln^p x}$, $p\in\mathbb{R}$.
14. $f(x)=e^x$.
15. $f(x)=e^{x^2}$.

4. Prove that the equation has a unique real root:

1. $x3^x=1$.
2. $x-\dfrac{1}{2}\sin x=\pi$.
3. $x^{13}+7x^3-5=0$.
4. $3^x+4^x=5^x$.
5. $2e^x+x^3+18x-6=0$.

5. Prove the following statements [4, 5, 16, 27-29, 48, 53]:

1. The equation $x^5+x^4+x^2+10x-5=0$ has a unique positive root lying in the interval $(0;1/2)$.

2. The equation $x\arcsin x=0$ has a unique real root $x=0$ of multiplicity $m=2$.

3. If the function $f:\mathbb{R}\to\mathbb{R}$ has a derivative on the interval $(-\infty;+\infty)$, then between two distinct zeros of the function f there lies at least one

zero of its derivative f'.

4. The equation $xe^x = 2$ has a unique positive root lying in the interval $(0;1)$.
5. If $a \in (1;+\infty)$ then the equation $a^x = ax$ has two real roots.
6. The equation $a^x = bx$ has two real roots if $a \in (1;+\infty)$ and $b > e\ln a$.
7. The equation $a^x = bx$ has no real roots if $e\ln a > b > 0$.
8. The equation $a^x = bx$ has a unique real root if $a \in (1;+\infty)$ and $b > 0$.
9. The equation $x\ln x = a$ has no real roots if $a \in (-\infty;-1/e)$.
10. The equation $x\ln x = a$ has a unique real root of multiplicity $m = 2$ if $a = -1/e$.
11. The equation $x\ln x = a$ has two simple roots if $a \in (-1/e;0)$.
12. The equation $x\ln x = a$ has a unique real root if $a \in (0;+\infty)$.

6. Find a function φ such that for the solution $x = x(y)$ of the equation $x + e^x = y$ holds $x = \varphi(y) + o(1)$ as $y \to \pm\infty$.

7. Find a function φ such that for the solution $x = x(y)$ of the equation $xe^x = y$ holds $x = \varphi(y) + o(1)$ as $y \to \pm\infty$.

8. Let x_{2k} be a root of the equation $\sin \pi x = 1/x$ lying in the intervals $(2k;1/2+2k)$, $k \in \mathbb{N}$. Prove that:

$$x_{2k} = 2k + \frac{1}{2\pi k} + o\left(\frac{1}{k^2}\right) \text{ as } k \to +\infty.$$

9. Let x_{2k+1} be a root of the equation $\sin \pi x = 1/x$ lying in the intervals $(1/2+2k;2(k+1))$, $k \in \mathbb{N}$. Prove that:

$$x_{2k+1} = 2k + 1 - \frac{1}{(2k+1)\pi} + o\left(\frac{1}{k^2}\right) \text{ as } k \to +\infty.$$

10. Let (x_n) with $x_1 < x_2 < \ldots$, be a sequence of positive solutions of the equation $\operatorname{tg} x = x$. Prove that:

$$x_n = (2n-1)\frac{\pi}{2} + O\left(\frac{1}{n}\right) \text{ as } n \to +\infty.$$

11. Determine the order and type of the function η:

1. $\eta(x) = 4x^3 + \ln(5+x)$.
2. $\eta(x) = \ln(1+e^{2x})$.
3. $\eta(x) = \ln(x+e^{4x})$.

4. $\eta(x)=\ln(x+e^{2x})+\ln(1+x^3)$.

5. $\eta(x)=\sqrt{x}+\ln(x^3+5x^4)$.

6. $\eta(x)=\ln(x^6+e^{2x})$.

7. $\eta(x)=\ln(1+e^{4x^3})$.

8. $\eta(x)=x\ln(1+x^{-1/2})$.

9. $\eta(x)=x\ln(1+x^2)\ln(1+x^{-3/2})$.

10. $\eta(x)=x\ln\left(1+\dfrac{1}{\ln(1+x)}\right)$.

11. $\eta(x)=4x^5+x\ln(1+e^x)$.

12. Prove the asymptotic formulas if the function $f:[0;+\infty)\to\mathbb{R}$ is continuous on $[0;+\infty)$ and $f(x)\sim x^\alpha$ as $x\to+\infty$:

1. $\int\limits_0^x f(t)dt\sim\dfrac{x^{\alpha+1}}{\alpha+1}$, $x\to+\infty$, $\alpha>-1$.

2. $\int\limits_0^x f(t)dt=O(1)$, $x\to+\infty$, $\alpha<-1$.

3. $\int\limits_0^x f(t)dt\sim\ln x$, $x\to+\infty$, $\alpha=-1$.

4. $\int\limits_x^{+\infty} f(t)dt\sim-\dfrac{x^{\alpha+1}}{\alpha+1}$, $x\to+\infty$, $\alpha<-1$.

13. Prove that if the function $\varphi:[0;+\infty)\to(0;+\infty)$ is continuously differentiable on $[0;+\infty)$ and $x\varphi'(x)=o\big(\varphi(x)\big)$ as $x\to+\infty$, then:

1. $\int\limits_0^x \varphi(t)t^{\alpha-1}dt\sim\dfrac{1}{\alpha}x^\alpha\varphi(x)$, $x\to+\infty$, $\alpha>0$.

2. $\int\limits_x^{+\infty} \varphi(t)t^{\alpha-1}dt\sim-\dfrac{1}{\alpha}x^\alpha\varphi(x)$, $x\to+\infty$, $\alpha<0$.

14. Prove the asymptotic formulas:

1. $\int\limits_1^x \dfrac{e^t}{t}dt=\dfrac{e^x}{x}+O\left(\dfrac{e^x}{x^2}\right)$, $x\to+\infty$.

2. $\sum\limits_{k=n+1}^{\infty}\dfrac{1}{k^2}\sim\dfrac{1}{n}$, $n\to\infty$.

3. $\sum_{k=1}^{n} k\ln k \sim \frac{1}{2}n^2\ln n$, $n\to\infty$.

4. $\sum_{k=n}^{\infty} \sqrt{k}e^{-\sqrt{k}} \sim 2ne^{-\sqrt{n}}$, $n\to\infty$.

5. $\int_0^x \sqrt[3]{1+t^3}dt = \frac{x^2}{2}+O(1)$, $x\to+\infty$.

6. $\int_0^x \ln^2 t dt = O\left(x\ln^2 x\right)$, $x\to 0+$.

7. $\sum_{k=1}^{n} k^2 \sim \frac{n^3}{3}$, $n\to\infty$.

1.14. Individual tasks.

1. Prove that:

1. $e^x \sim 1+x$, $x\to 0$.
2. $e^x + x^3 = O(e^x)$, $x\in[0;+\infty)$.
3. $x\sin\frac{1}{x} = O\left(|x|\right)$, $x\to 0$.
4. $\frac{\operatorname{arctg} x}{1+x^2} = O\left(\frac{1}{x^2}\right)$, $x\to+\infty$.
5. $\int_{x^2}^{x^3} \sin t^2 = o(x)$, $x\to 0$.
6. $\sqrt{x+\sqrt{x+\sqrt{x}}} \sim \sqrt[8]{x}$, $x\to 0$.
7. $x^2 + x\ln^{100} x \sim x^2$, $x\to+\infty$.
8. $\ln(1+2x) \asymp x$, $x\to 0$.
9. $\ln\left(1+\sin\left(\frac{2x}{e^x}\right)\right) \asymp x$, $x\to 0$.
10. $\sqrt{1+x^2} - x = O\left(\frac{1}{x}\right)$, $x\to+\infty$.
11. $x\sin\sqrt{x} = O(x^{3/2})$, $x\to 0$.
12. $\operatorname{arctg}\frac{1}{x} = O(1)$, $x\to 0$.
13. $(1+x)^n = 1+nx+o(x)$, $x\to 0$.

14. $x+x^2\sin x=O(x^2)$, $x\to+\infty$.

15. $\ln x=o(x^{\alpha})$, $x\to+\infty$, $\alpha>0$.

16. $\ln\ln x=o(\ln x)$, $x\to+\infty$.

17. $2x+\ln x+\sin x=O(x)$, $x\to+\infty$.

18. $\sqrt{x^2+x+1}-x\sim\frac{1}{2}$, $x\to+\infty$.

19. $\frac{1}{x}+\frac{1}{x^3}+\frac{1}{x^5}=O\left(\frac{1}{x}\right)$, $x\to+\infty$.

20. $e^x-e=e(x-1)+o(x-1)$, $x\to1$.

21. $\sqrt{1+x}-\sqrt{1-x}=x+o(x)$, $x\to0$.

22. $\frac{2x^5}{x^3-3x+1}=2x^2+o(x^2)$, $x\to+\infty$.

23. $\sqrt{x^2+\sqrt{x^3+\sqrt{x^5}}}=O(x)$, $x\to+\infty$.

24. $e^{\sin x}=1+x+\frac{1}{2}x^2+o(x^2)$, $x\to0$.

25. $\operatorname{arctg}x=x-\frac{1}{3}x^3+o(x^3)$, $x\to0$.

26. $e^{\operatorname{tg}x}=1+x+o(x)$, $x\to0$.

27. $\frac{1+x}{1+x^2}=O\left(\frac{1}{x}\right)$, $x\to+\infty$.

28. $\operatorname{sh}x=O(e^x)$, $x\to+\infty$.

29. $\ln x=o(x^{-\alpha})$, $x\to0+$, $\alpha>0$.

30. $n!\sim\sqrt{2\pi n}e^{-n}n^n$, $n\to\infty$.

2. Compare the infinitesimal functions f and φ at a given point a and determine whether they are equivalent, of the same order, whether one is of higher order than the other, or whether they are incomparable:

1. $f(x)=e^{\sin x}-e^x$, $\varphi(x)=x^3$, $a=0$.
2. $f(x)=x-\sin x, \varphi(x)=e^x-1-x-\frac{x^2}{2}$, $a=0$.
3. $f(x)=x^x-x$, $\varphi(x)=(x-1)^2$, $a=1$.
4. $f(x)=4(1-x^4)-5(1-x^5)$, $\varphi(x)=(1-x)^2$, $a=1$.
5. $f(x)=\ln x-x+1$, $\varphi(x)=(x-1)\ln x$, $a=1$.

6. $f(x)=2^x-x^2$, $\varphi(x)=x-2$, $a=2$.

7. $f(x)=1-\sqrt{1+x^2}\cos x$, $\varphi(x)=\sin^4 x$, $a=0$.

8. $f(x)=\cos x-e^{-x^2/2}$, $\varphi(x)=x^4$, $a=0$.

9. $f(x)=\arcsin 2x-2\arcsin x$, $\varphi(x)=x^2$, $a=0$.

10. $f(x)=\dfrac{1}{x^2}-\operatorname{tg}^2\dfrac{1}{x}$, $\varphi(x)=\dfrac{1}{x^4}$, $a=+\infty$.

11. $f(x)=e^x-1-x-\dfrac{x^2}{2}-\dfrac{x^3}{6}$, $\varphi(x)=x^4$, $a=0$.

12. $f(x)=\dfrac{\pi}{2}-\operatorname{arctg}\sqrt{x}$, $\varphi(x)=\dfrac{1}{\sqrt{x}}$, $a=+\infty$.

13. $f(x)=e^x-1-x-\dfrac{x^2}{2}-\dfrac{x^3}{6}$, $\varphi(x)=x^4$, $a=0$.

14. $f(x)=\sin 2x^2$, $\varphi(x)=x^2$, $a=0$.

15. $f(x)=x-\operatorname{tg} x$, $\varphi(x)=x^3$, $a=0$.

16. $f(x)=\sqrt{\pi}-\sqrt{\arccos x}$, $\varphi(x)=\sqrt{x+1}$, $a=-1$.

17. $f(x)=e^{x^2}-e^{-x^2}$, $\varphi(x)=\ln(1+x^2)$, $a=0$.

18. $f(x)=x^x-x$, $\varphi(x)=(x-1)^2$, $a=1$.

19. $f(x)=\operatorname{tg}\dfrac{\pi x}{x^2+1}$, $\varphi(x)=\dfrac{1}{x}$, $a=+\infty$.

20. $f(x)=e^x-\sum\limits_{k=0}^{7}\dfrac{x^k}{k!}$, $\varphi(x)=x^8$, $a=0$.

21. $f(x)=\arcsin 2x-\sin^2 x$, $\varphi(x)=x^2+\ln(1+3x)$, $a=0$.

22. $f(x)=\cos 3x-\cos x$, $\varphi(x)=x^2$, $a=0$.

23. $f(x)=\sqrt{9-x}-3$, $\varphi(x)=x$, $a=0$.

24. $f(x)=\operatorname{tg} x-\sin x$, $\varphi(x)=x^3$, $a=0$.

25. $f(x)=\ln(1+x^2)$, $\varphi(x)=\cos x-e^{-x^2}$, $a=0$.

26. $f(x)=x-(1+x)\ln(1+x)$, $\varphi(x)=x^2(1+x)$, $a=0$.

27. $f(x)=x^3-3^x$, $\varphi(x)=x-3$, $a=3$.

28. $f(x)=(1-x^4)(1-x^5)$, $\varphi(x)=(1-x)^3$, $a=1$.

29. $f(x)=e^{x}+e^{-x}-2$, $\varphi(x)=x^{2}$, $a=0$.

30. $f(x)=x^{3}-3^{x}$, $\varphi(x)=(x-3)^{2}$, $a=3$.

3. Find the numbers a_j for which the asymptotic formulas are valid:

1. $x^{x}-1=a_1(x-1)+a_2(x-1)^{2}+a_3(x-1)^{3}+o\left((x-1)^{3}\right)$, $x\to 1$.

2. $\int_0^{+\infty} e^{-tx}\sin t\,dt=\frac{a_0}{x}+\frac{a_1}{x^{2}}+o\left(\frac{1}{x^{2}}\right)$, $x\to+\infty$.

3. $\frac{x^{x}-x}{(x-1)^{2}}=a_0+a_1(x-1)+o(x-1)$, $x\to 1$.

4. $\int_0^{+\infty} e^{-tx}(1+t)^{10}\,dt=\frac{a_0}{x}+\frac{a_1}{x^{2}}+o\left(\frac{1}{x^{2}}\right)$, $x\to+\infty$.

5. $\ln\sin x=a_0+a_1(x-\pi/2)+a_2(x-\pi/2)^{2}+a_3(x-\pi/2)^{3}+$

$$+a_4(x-\pi/2)^{4}+o\left((x-\pi/2)^{4}\right),\ x\to\pi/2.$$

6. $\int_0^{+\infty} e^{-tx}\sqrt{1+t}\,dt=\frac{a_0}{x}+\frac{a_1}{x^{2}}+o\left(\frac{1}{x^{2}}\right)$, $x\to+\infty$.

7. $\frac{\sin x}{1-2x}=a_0+a_1x+a_2x^{2}+o(x^{2})$, $x\to 0$.

8. $\int_0^{+\infty} e^{-tx}\sqrt{1+t^{2}}\,dt=\frac{a_0}{x}+\frac{a_1}{x^{2}}+o\left(\frac{1}{x^{2}}\right)$, $x\to+\infty$.

9. $\sin(\sin x)=a_0+a_1x+a_2x^{2}+o(x^{2})$, $x\to 0$.

10. $\int_0^{+\infty} e^{-tx}\frac{1}{1+t}\,dt=\frac{a_0}{x}+\frac{a_1}{x^{2}}+\frac{a_2}{x^{3}}+o\left(\frac{1}{x^{3}}\right)$, $x\to+\infty$.

11. $\ln\left(1+\sin x\right)=a_0+a_1x+o(x)$, $x\to 0$.

12. $\int_0^{+\infty} e^{-tx}\frac{1}{(1+t)^{2}}\,dt=\frac{a_0}{x}+\frac{a_1}{x^{2}}+o\left(\frac{1}{x^{2}}\right)$, $x\to+\infty$.

13. $\operatorname{tg} x=a_0+a_1x+a_2x^{2}+o(x^{2})$, $x\to 0$.

14. $\operatorname{arctg} x=a_0+a_1\frac{1}{x}+o\left(\frac{1}{x}\right)$, $x\to+\infty$.

15. $\operatorname{arctg} x=a_0+a_1\frac{1}{x}+o\left(\frac{1}{x}\right)$, $x\to-\infty$.

16. $\arcsin x = a_0 + a_1(x-1) + a_2(x-1)^2 + o((x-1)^2)$, $x \to 1-$.

17. $\dfrac{1}{2+x} = a_0 + a_1(1+x) + a_2(1+x)^2 + o((1+x)^2)$, $x \to -1$.

18. $e^{x\sin x} = a_0 + a_1 x + o(x)$, $x \to 0$.

19. $\sin(x^2 - 2x + 3) = a_0 + a_1(x-1) + o(x-1)$, $x \to -1$.

20. $\dfrac{x}{1+x} = a_0 + a_1(x-1) + o(x-1)$, $x \to 1$.

21. $\dfrac{1-3x}{2+x} = a_0 + a_1(1+x) + a_2(1+x)^2 + O((1+x)^3)$, $x \to -1$.

22. $\dfrac{\cos x}{1+2x} = a_0 + a_1 x + a_2 x^2 + O(x^3)$, $x \to 0$.

23. $\dfrac{(1+x)^2}{1-x} = a_0 + a_1 x + a_2 x^2 + o(x^2)$, $x \to 0$.

24. $\cos(x^2 + x - 1) = a_0 + a_1(x-1) + a_2(x-1)^2 + o((x-1)^2)$, $x \to 1$.

25. $\dfrac{e^x - 1}{1-x} = a_0 + a_1 x + a_2 x^2 + o(x^2)$, $x \to 0$.

26. $\sin\left(\ln(1+x)\right) = a_0 + a_1 x + o(x)$, $x \to 0$.

27. $\dfrac{1+x+x^2}{1-x+x^2} = a_0 + a_1 x + a_2 x^2 + o(x^2)$, $x \to 0$.

28. $\dfrac{x}{1+x} = a_0 + a_1(x-1) + a_2(x-1)^2 + o((x-1)^2)$, $x \to 1$.

29. $\dfrac{\cos x}{1+2x} = a_0 + a_1 x + a_2 x^2 + O(x^3)$, $x \to 0$.

30. $\ln(1+2x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + O(x^4)$, $x \to 0$.

4. Justify the asymptotic formulas:

1. $\displaystyle\int_1^x e^{t^2}\,dt \sim \frac{e^{x^2}}{2x}$, $x \to +\infty$.

2. $\displaystyle\int_x^{+\infty} e^{-t^2}\,dt \sim \frac{e^{-x^2}}{2x}$, $x \to +\infty$.

3. $\displaystyle\sum_{k=2}^{n} \frac{\ln k}{k} \sim \frac{\ln^2 n}{2}$, $n \to \infty$.

4. $\sum_{k=1}^{n}\frac{1}{k^{1/3}}\sim\frac{3n^{2/3}}{2},\ n\to\infty$.

5. $\sum_{k=1}^{n}\frac{1}{\sqrt{k}}\sim 2\sqrt{n},\ n\to\infty$.

6. $\int_{x}^{+\infty}\frac{\cos t}{t^2}dt=-\frac{\sin x}{x^2}+O\left(\frac{1}{x^3}\right),\ x\to+\infty$.

7. $\int_{x}^{+\infty}\frac{\sin t}{t}dt=\frac{\sin x}{x}+O\left(\frac{1}{x^2}\right),\ x\to+\infty$.

8. $\int_{1}^{x}e^{t^2}dt=\frac{e^{x^2}}{2x}-\frac{e^{x^2}}{4x^2}+o\left(\frac{e^{x^2}}{x^2}\right),\ x\to+\infty$.

9. $\int_{1}^{x}\frac{e^t}{t}dt=\frac{e^x}{x}+\frac{e^x}{x^2}+o\left(\frac{e^x}{x^2}\right),\ x\to+\infty$.

10. $\sum_{k=n}^{\infty}\frac{1}{k^5}\sim\frac{1}{4n^4},\ n\to\infty$.

11. $\sum_{n^2\le x}\ln n=((1+o(1))\sqrt{x}\ln\sqrt{x},\ x\to+\infty$.

12. $\int_{1}^{x}\left(1+\frac{1}{t}\right)^t dt=ex-\frac{1}{2}e\ln x+O(1),\ x\to+\infty$.

13. $\sum_{k=n}^{\infty}k^2e^{-k}\sim\frac{e}{e-1}n^3e^{-n},\ n\to\infty$.

14. $\int_{0}^{x}\sqrt{1+t^2}dt=x+\frac{1}{2}\ln x+O(1),\ x\to+\infty$.

15. $\int_{x}^{+\infty}\frac{dt}{t\sqrt{t^2+t+2}}=\frac{1}{x}+O\left(\frac{1}{x^2}\right),\ x\in[1;+\infty)$.

16. $\sum_{k=1}^{n}\frac{\ln k}{k}\sim\frac{1}{2}\ln^2 n,\ n\to\infty$.

17. $\int_{0}^{+\infty}\frac{e^{-t}}{t+x}dt\sim\sum_{k=0}^{\infty}(-1)^k\frac{k!}{x^{k+1}},\ x\to+\infty$.

18. $\int_{0}^{+\infty}e^{-x\operatorname{sh}t}dt\sim\frac{1}{x}-\frac{1}{x^3}+\frac{(1\cdot 3)^2}{x^5}-\frac{(1\cdot 3\cdot 5)^2}{x^7}+\ldots,\ x\to+\infty$.

19. $\int\limits_{x}^{+\infty} e^{-t^2}dt \sim \frac{e^{-x^2}}{2x}\sum_{k=0}^{\infty}(-1)^k\frac{(2k-1)!!}{2^k x^{2k}},\ x\to+\infty.$

20. $\frac{1}{\pi}\int\limits_{0}^{\pi} e^{x\cos t}\cos(nt)dt \sim \frac{e^{-x}}{\sqrt{2\pi x}},\ x\to+\infty.$

21. $\frac{1}{\pi}\int\limits_{0}^{\pi}\cos(x\sin t-nt)dt = \sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{\pi}{4}-\frac{\pi n}{2}\right)+O\left(\frac{1}{x}\right),\ x\to+\infty.$

22. $\int\limits_{0}^{1}\frac{\arcsin t}{\sqrt{1-t^2}}\sin(xt)dt = \sqrt{\frac{\pi}{2x}}\sin\left(x-\frac{\pi}{4}\right)\left(\frac{\pi}{2}+O\left(\frac{1}{x}\right)\right),\ x\to+\infty.$

23. $\int\limits_{0}^{x}\frac{\cos t-1}{t}dt = \sum_{k=1}^{\infty}\frac{(-1)^k}{2k}\cdot\frac{x^{2k}}{(2k)!},\ x\to 0.$

24. $\int\limits_{0}^{x}\frac{\sin t}{t}dt = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cdot\frac{x^{2k+1}}{(2k+1)!},\ x\to 0.$

25. $\int\limits_{x}^{+\infty}\frac{e^{it}}{\sqrt{t}}dt \sim \frac{ie^{ix}}{\sqrt{x}}\sum_{k=0}^{\infty}\frac{\Gamma(k+1/2)}{\sqrt{\pi}(ix)^k},\ x\to+\infty.$

26. $\int\limits_{0}^{+\infty} e^{-xt}\ln(1+t^2)dt \sim \sum_{k=1}^{\infty}(-1)^{k-1}\frac{(2k)!}{kx^{2k+1}},\ x\to+\infty.$

27. $\int\limits_{0}^{\pi/2}\sin^n t dt \sim \sqrt{\frac{\pi}{2n}},\ n\to+\infty.$

28. $\int\limits_{-1}^{1}(1-t^2)^n dt \sim \sqrt{\frac{\pi}{n}},\ n\to+\infty.$

29. $\int\limits_{0}^{1} e^t t^n (1+t^2)^{-n} dt \sim \sqrt{\frac{\pi}{2n}}\cdot\frac{e}{2^n},\ n\to+\infty.$

30. $\int\limits_{0}^{\pi/2} e^{-x\,\mathrm{tg}\,t}dt \sim \frac{1}{x}\sum_{k=0}^{\infty}(-1)^k\frac{(2k)!}{x^{2k}},\ x\to+\infty.$

Chapter 2. The simplest properties of entire functions

2.1. The maximum of the modulus of an entire function. Entire transcendental functions. A function $f:\mathbb{C}\to\mathbb{C}$ is called holomorphic [3, 26] at a point a if it has a derivative in some neighborhood of that point a. A function f is called holomorphic on a set E if it is holomorphic at every point of that set. A function f that is holomorphic at every point $z=x+iy\in\mathbb{C}$ is called an entire function [22, 24, 28, 31, 45, 47, 51, 54]. Thus, an entire function is a function f that has a derivative at every point $z\in\mathbb{C}$. For a function f to have a derivative at a point $z=x+iy$, it is necessary and sufficient that the functions $u=\mathrm{Re}\, f$ and $v=\mathrm{Im}\, f$, treated as functions of two variables, are differentiable at the point $(x;y)$ and satisfy at that point the Cauchy-Riemann conditions [3, 26]:

$$\begin{cases} \dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}, \\ \dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}. \end{cases}$$

In order that a function f be an entire function, it is necessary and sufficient that it represented by a power series of the form [1, 22, 24, 28, 31, 45, 47, 51, 54]

$$f(z)=\sum_{k=0}^{\infty} f_k z^k \tag{1}$$

that converges for all $z\in\mathbb{C}$. The series (1) converges for all $z\in\mathbb{C}$ if and only if $\lim\limits_{k\to\infty}\sqrt[k]{|f_k|}=0$. The Taylor coefficients f_k of an entire function f are determined by the formulas [1, 22, 24, 28, 31, 45, 47, 51, 54]:

$$f_k=\frac{f^{(k)}(0)}{k!},\ \ f_k=\frac{1}{2\pi i}\int\limits_{\partial U(0;R)}\frac{f(t)}{t^{k+1}}dt.$$

From the last formula, follow the Cauchy inequalities: $(\forall r>0)(\forall k\geq 0):|f_k|r^k\leq M_f(r)$, where $M_f(r)=\max\{|f(z)|:|z|\leq r\}$ is the maximum of the modulus of an entire function f. The maximum of the modulus is one of the key characteristics of an entire function. It follows from the maximum modulus principle that [1, 22, 24, 28, 31, 45, 47, 51, 54]:

$$M_f(r)=\max\{|f(z)|:|z|=r\}=\max\left\{\left|f(re^{i\theta})\right|:\theta\in[0;2\pi]\right\}.$$

Finding the maximum of the modulus of an entire function is possible only in specific cases.

Example 1. *The functions* $f(z)=e^z$, $f(z)=\sin z$, $f(z)=\cos z$ $f(z)=e^{\sin z}$ *and* $f(z)=e^{z^\rho}$, *where* $\rho\in\mathbb{Z}_+$, *are entire functions. The functions* $f(z)=\sqrt{z}$, $f(z)=\operatorname{Ln} z$ *and* $f(z)=e^{z^\rho}$ *with* $\rho\notin\mathbb{Z}_+$, *are not entire.*

Example 2. *If* $f(z)=1+z$, *then* $\left|f(re^{i\theta})\right|=\sqrt{1+2r\cos\theta+r^2}$ *and* $M_f(r)=1+r$.

Example 3. *If* $f(z)=e^z$, *then* $\left|f(re^{i\theta})\right|=\left|e^{r(\cos\theta+i\sin\theta)}\right|=e^{r\cos\theta}$ *and* $M_f(r)=e^r$.

Example 4. *If* $f(z)=e^{\tau z^n}$, *where* $\tau=se^{i\psi}\in\mathbb{C}$ *and* $n\in\mathbb{N}$, *then*

$$\left|f(re^{i\theta})\right|=\left|e^{se^{i\psi}r^ne^{in\theta}}\right|=\left|e^{sr^ne^{i(n\theta+\psi)}}\right|=\left|e^{sr^n(\cos(n\theta+\psi)+i\sin(n\theta+\psi))}\right|=$$

$$=e^{sr^n\cos(n\theta+\psi)}\left|\left(\cos\left(sr^n\sin(n\theta+\psi)\right)+i\sin\left(sr^n\sin(n\theta+\psi)\right)\right)\right|=e^{sr^n\cos(n\theta+\psi)}.$$

Therefore, $M_f(r)=e^{sr^n}$.

Example 5. *If* $f(z)=e^z+z$, *then* $|f(z)|=|e^z+z|\le|e^z|+|z|\le e^r+r$ *for* $|z|\le r$. *Thus* $M_f(r)\le e^r+r$. *On the other hand,* $|f(r)|=|e^r+r|=e^r+r$ *and* $M_f(r)\ge e^r+r$. *Hence,* $M_f(r)=e^r+r$.

Example 6. *If* $f(z)=\sum_{k=0}^{n}f_kz^k$ *is a polynomial of degree* n, *then* $|f(z)|=|f_n||z|^n(1+o(1))$ *as* $z\to\infty$, *and* $M_f(r)=(1+o(1))|f_n|r^n$ *as* $r\to+\infty$. *Thus, the degree of a polynomial can be determined using the formula (here* $\ln^+x:=\max\{0;\ln x\}$ *for* $x>0$*):*

$$n=\lim_{r\to+\infty}\frac{\ln^+M_f(r)}{\ln r}.$$

Example 7. *If* f *is an entire function, then* $M_f(r)\le\sum_{k=0}^{\infty}|f_k|r^k$, *because*

$$M_f(r)=\max_{\theta\in[0;2\pi]}\left|\sum_{k=0}^{\infty} f_k r^k e^{ik\theta}\right|\le \max_{\theta\in[0;2\pi]}\sum_{k=0}^{\infty}|f_k|r^k=\sum_{k=0}^{\infty}|f_k|r^k .$$

Example 8. *If* f *is an entire function and all* $f_k\ge 0$, *then* $M_f(r)=\sum_{k=0}^{\infty} f_k r^k$. *Indeed,* $M_f(r)\ge|f(r)|=\sum_{k=0}^{\infty} f_k r^k$. *On the other hand,* $M_f(r)\le\sum_{k=0}^{\infty}|f_k|r^k=\sum_{k=0}^{\infty} f_k r^k$.

Example 9. *Let* $f(z)=\sin z$ *and* $m_f(r)=\min\{|f(z)|:|z|=r\}$. *Then* $M_f(r)=(e^r-e^{-r})/2$ *and* $m_f(r)=|\sin r|$.

Indeed, $|\sin z|=\sqrt{\operatorname{ch}^2 y-\cos^2 x}$. *If* $x^2+y^2=r^2$, *then* $\operatorname{ch}^2 y-\cos^2 x=\operatorname{ch}^2\sqrt{r^2-x^2}-\cos^2 x$. *But*

$$(\operatorname{ch}^2\sqrt{r^2-x^2}-\cos^2 x)'_x=-\frac{x\operatorname{sh}2\sqrt{r^2-x^2}}{\sqrt{r^2-x^2}}+\sin 2x,$$

$$\frac{\operatorname{sh}u}{u}>1,\ u\in(0;+\infty),\ \frac{\sin x}{x}<1,\ x\in(0;+\infty).$$

Therefore, $(\operatorname{ch}^2\sqrt{r^2-x^2}-\cos^2 x)'_x\le 0$ *for all* $x\in[0;r]$. *Hence,*

$$m_f(r)=|f(r)|=\sqrt{1-\cos^2 r}=|\sin r| \text{ and } M_f(r)=|f(ir)|=(e^r-e^{-r})/2.$$

Every polynomial is an entire function. An entire function that is not a polynomial is called an entire transcendental function. Entire transcendental functions can be thought of as polynomials of infinite degree.

Theorem 1 (Liouville). *For an entire function* f *to be a polynomial, it is necessary and sufficient that*

$$\varliminf_{r\to+\infty}\frac{\ln^+ M_f(r)}{\ln r}<+\infty. \qquad (2)$$

Proof. If $f(z)=\sum_{k=0}^{n} f_k z^k$ is a polynomial, then $\ln^+ M_f(r)=(1+o(1))n\ln r$ as $r\to+\infty$, and the condition (2) holds. Conversely, suppose condition (2) is satisfied. Then, there exist a number c_1 and a sequence (r_n), $0<r_n\uparrow+\infty$, such that $M_f(r_n)\le c_1 r_n^{c_1}$. Therefore, using Cauchy's inequalities for all $k\in\mathbb{Z}_+$ and $n\in\mathbb{N}$, we obtain $|f_k|\le c_1 r_n^{k-c_1}$.

Taking the limit as n to $+\infty$, we conclude that $|f_k| = 0$ for $k > c_1$, that is f is a polynomial. The theorem is proved. ►

Corollary 1. *For an entire function f to be an entire transcendental function, it is necessary and sufficient that*

$$\lim_{r\to+\infty} \frac{\ln^+ M_f(r)}{\ln r} = +\infty .$$

Theorem 2 (Hadamard's Three Circle Theorem). *Let $f \not\equiv 0$ be an entire function Then, for any r_1, r and r_2, $0 < r_1 < r < r_2$, the following inequality holds:*

$$\ln M_f(r) \le \ln M_f(r_1)\frac{\ln r_2 - \ln r}{\ln r_2 - \ln r_1} + \ln M_f(r_2)\frac{\ln r - \ln r_1}{\ln r_2 - \ln r_1} .$$

Proof. Let $D = \{z : r_1 \le |r| \le r_2\}$ and $g(z) = z^\alpha f(z)$, $\alpha \in \mathbb{R}$. Then $|g(z)|$ is a single-valued function in $\overline{D}$. By the maximum modulus principle, the function g attains its maximum and minimum values in $\overline{D}$, which must be attained on the boundary ∂D. Hence,

$$r^\alpha M_f(r) \le \max\left\{r_1^\alpha M_f(r_1); r_2^\alpha M_f(r_2)\right\} . \tag{3}$$

Let us choose α such that $r_1^\alpha M_f(r_1) = r_2^\alpha M_f(r_2)$. Then, from (3), we obtain the required inequality. The theorem is proved. ►

Corollary 2 ([1, 22, 24, 28, 31, 45, 47, 51, 54]). *The function $\ln M_f(r)$ is convex with respect to $\ln r$ on $(0;+\infty)$.*

Corollary 3 ([1, 22, 24, 28, 31, 45, 47, 51, 54]). *The right-hand derivative $\beta_f(r) = \dfrac{d\ln M_f(r)}{d\ln r}$ is a non-decreasing function and for every entire function $f \not\equiv 0$, we have*

$$\ln M_f(r_2) - \ln M_f(r_1) = \int_{r_1}^{r_2} \beta_f(r)\, d\ln r, \quad 0 < r_1 < r_2 .$$

Corollary 4. *If f is an entire transcendental function, then $\beta_f(r) \to +\infty$ as $r \to +\infty$.*

Corollary 5 ([1, 22, 24, 28, 31, 45, 47, 51, 54]). *If f is an entire transcendental function, then there exists a function $u : (0;+\infty) \to (0;+\infty)$ such* *that* $u(r)\to+\infty$ *and* $\exp(u(r))M_f(r)=M_f((1+o(1))r)$ *as* $r\to+\infty$. *One can take, for example,* $u(r)=\sqrt{\beta_f(r)}$, *because*

$$\sqrt{\beta_f(r)}+\ln M_f(r)=\ln M_f((1+\varepsilon)r)-\int_r^{(1+\varepsilon)r}\beta_f(t)dt+\sqrt{\beta_f(r)}\le$$

$$\le\ln M_f((1+\varepsilon)r)-\beta_f(r)\ln(1+\varepsilon)+\sqrt{\beta_f(r)}\le\ln M_f((1+\varepsilon)r)$$

for any $\varepsilon>0$ *if* $r\ge r_0(\varepsilon)$.

Corollary 6 ([1, 22, 24, 28, 31, 45, 47, 51, 54]). *If* f *is an entire transcendental function and* Ψ_f^{-1} *is the function inverse to* $\Psi_f(r)=\ln M_f(r)$, *then*

$$\lim_{x\to\infty}\frac{\Psi_f^{-1}(x+O(1))}{\Psi_f^{-1}(x)}=1.$$

2.2. The maximal term and central index of an entire function. Inequalities between the maximum of the modulus and the maximal term of an entire function. Let f be an entire transcendental function with Taylor coefficients f_k. The functions $\mu_f(r)=\max\{|f_k|r^k:k\ge0\}$ and $\nu_f(r)=\{k:\mu_f(r)=|f_k|r^k\}$ are called, respectively, the maximal term and the central index of an entire function f [1, 22, 24, 28, 31, 45, 47, 51, 54]. Since f is an entire function, it holds that $\sqrt[k]{|f_k|}\to0$ as $k\to\infty$, and therefore $|f_k|r^k\to0$ as $k\to\infty$ for every $r>0$. Hence, $\mu_f(r)<+\infty$ and $\nu_f(r)<+\infty$ for each $r\in[0;+\infty)$.

Theorem 1 ([1, 22, 24, 28, 31, 45, 47, 51, 54]). *For every entire function* f *holds*

$$\mu_f(r)\le M_f(r)\le(1+1/\varepsilon)\mu_f((1+\varepsilon)r),$$

for all $r\ge0$ *and* $\varepsilon>0$.

Proof. Indeed, the left-hand side follows from the Cauchy inequalities $|f_n|r^n\le M_f(r)$, while the right-hand side follows from the inequalities

$$M_f(r)\le\sum_{k=0}^{\infty}|f_k|r^k\le\sum_{k=0}^{\infty}|f_k|(r(1+\varepsilon))^k(1+\varepsilon)^{-k}\le\mu_f((1+\varepsilon)r)(1+1/\varepsilon).$$

The theorem is proved. ►

Theorem 2 ([1, 22, 24, 28, 31, 45, 47, 51, 54]). *If* $\chi_0(f)=0$, $\chi_n(f)=|f_{n-1}/f_n|$ *for* $n\in\mathbb{N}$ *and*

$$0<\chi_1(f)\le\chi_2(f)\le\dots, \qquad (1)$$

then $\mu_f(r)=|f_m|r^m$ *for* $r\in[\chi_m(f);\chi_{m+1}(f)]$ *and* $\nu_f(r)=m$ *if* $r\in[\chi_m(f);\chi_{m+1}(f))$.

Proof. Indeed, $f_m=f_0/(\chi_1(f)\dots\chi_m(f))$. Therefore, for $r\in[\chi_m(f);\chi_{m+1}(f)]$

$$\frac{|f_k|r^k}{|f_m|r^m}=\chi_{k+1}(f)\dots\chi_m(f)/r^{m-k}\le\left(\chi_m(f)/r\right)^{m-k}\le 1,\ k<m,$$

$$\frac{|f_k|r^k}{|f_m|r^m}=r^{k-m}/\chi_{k+1}(f)\dots\chi_m(f)\le\left(r/\chi_{m+1}(f)\right)^{k-m}\le 1,\ k>m.$$

In addition,

$$\frac{|f_k|r^k}{|f_m|r^m}=r^{k-m}/\chi_{k+1}(f)\dots\chi_m(f)<\left(r/\chi_{m+1}(f)\right)^{k-m}\le 1,\ k>m,$$

if $r\in[\chi_m(f);\chi_{m+1}(f)]$ and $k>m$. Theorem 2 is proved. ►

Example 1. *If* $f(z)=e^z$, *then* $f_k=1/k!$ *and* $\chi_k=k$. *Therefore,* $\mu_f(r)=r^m/m!$ *for* $r\in[m;m+1]$. *Thus, using Stirling's formula*

$$m!=(1+o(1))\sqrt{2\pi m}(m/e)^m,\ m\to\infty,$$

we obtain

$$\mu_f(r)=(1+o(1))(re/m)^m/\sqrt{2\pi r},\ r\to+\infty.$$

The function $\varphi(t)=(re/t)^t$ *on the segment* $[r-1;r]$ *is increasing,* $\varphi(r)=e^r$ *and*

$$\varphi(r-1)=\left(er/(r-1)\right)^{r-1}=(1+o(1))e^r,\ r\to+\infty.$$

Hence,

$$\mu_f(r)=(1+o(1))e^r/\sqrt{2\pi r},\ r\to+\infty.$$

Theorem 3 ([1, 22, 24, 28, 31, 45, 47, 51, 54]). *If* f *is an entire transcendental function, then* ν_f *is a non-decreasing function and*

$$\ln\mu_f(r)=\ln\mu_f(r_0)+\int_{r_0}^{r}\frac{\nu_f(t)}{t}dt,\ 0<r_0<r. \qquad (2)$$

Proof. Indeed, continuity on the right and monotonicity follow from the previous theorem. Further, for $h>0$, we have

$$\ln\mu_f(r+h)=\ln\left|a_{\nu_f(r+h)}(r+h)^{\nu_f(r+h)}\right|\le\ln\left(\mu_f(r)\left(\frac{r+h}{r}\right)^{\nu_f(r+h)}\right)\le$$

$$\le\ln\mu_f(r)+\nu_f(r+h)\ln\left(1+\frac{h}{r}\right),$$

$$\ln\mu_f(r)=\ln\left|f_{\nu_f(r)}r^{\nu_f(r)}\right|\le\ln\left(\mu_f(r+h)\left(\frac{r}{r+h}\right)^{\nu_f(r)}\right)=$$

$$=\ln\mu_f(r+h)-\nu_f(r)\ln\left(1+\frac{h}{r}\right),$$

$$\nu_f(r)\le\frac{\ln\mu_f(r+h)-\ln\mu_f(r)}{\ln\left(1+\frac{h}{r}\right)}\le\nu_f(r+h)\,. \tag{3}$$

Therefore, the function ν_f is non-decreasing and, hence, continuous everywhere on $[0;+\infty)$ except possibly at a countable set of points. Thus, by writing the analog of the inequalities (3) for $h<0$, for all $t>0$ except possibly for a set of measure zero, we obtain

$$\left(\ln\mu_f(t)\right)'=\frac{\nu_f(t)}{t}\,. \tag{4}$$

Since the functions $\ln\left(|f_k|r^k\right)$ are convex with respect to $\ln r$, the function $\ln\mu_f(r)$ is also convex with respect to $\ln r$. Hence, the function $\ln\mu_f(r)$ is absolutely continuous on every finite interval. Therefore, from (4) it follows (2). The theorem is proved. ►

Corollary 1. *If* $f_0=1$ *and* $0<\chi_1(f)\le\chi_2(f)\le\ldots$, *then*

$$\ln\mu_f(r)=\int_0^r\frac{\nu_f(t)}{t}dt=\int_0^r\ln\frac{r}{t}d\nu_f(t)=\sum_{0<\chi_k(f)\le r}\ln\frac{r}{\chi_k(f)}=\ln\frac{r^{\nu_f(r)}}{\chi_1(f)\ldots\chi_{\nu_f}(f)}\,.$$

Theorem 4 ([1, 22, 24, 28, 31, 45, 47, 51, 54]). *For every entire function* f *holds*

$$M_f(r)\le\mu_f(r)\left(\nu_f(r)+(1+\varepsilon)^{\nu_f((1+\varepsilon)r)-\nu_f(r)-p+1}/\varepsilon\right),$$

for all $r\ge0$, $\varepsilon>0$ *and* $p\in\mathbb{N}$.

Proof. Indeed,

$$M_f(r)\le\sum_{k=0}^{\nu_f(r)+p-1}|f_k|r^k+\sum_{k=\nu_f(r)+p}^{\infty}|f_k|(r(1+\varepsilon))^k(1+\varepsilon)^{-k}\le$$

$$\le\mu_f(r)\left(\nu_f(r)+p\right)+\mu_f((1+\varepsilon)r)(1+\varepsilon)^{-\nu_f(r)-p+1}/\varepsilon\,.$$

In addition,

$$\ln\mu_f((1+\varepsilon)r)=\ln\mu_f(r)+\int_r^{(1+\varepsilon)r}\frac{\nu_f(t)}{t}dt\le\nu_f((1+\varepsilon)r)\ln(1+\varepsilon)\,,$$

and we obtain the required inequality. Theorem 4 is proved. ►

Theorem 5 ([1, 22, 24, 28, 31, 45, 47, 51, 54]). *For every entire function f holds*

$$M_f(r)\le\mu_f(r)\left(\nu_f((1+\varepsilon)r)+p+(1+\varepsilon)^{-p+1}/\varepsilon\right),$$

for all $r\ge 0$, $\varepsilon>0$ and $p\in\mathbb{N}$.

Proof. We have

$$M_f(r)\le\sum_{k=0}^{\nu_f((1+\varepsilon)r)+p-1}|f_k|r^k+\sum_{k=\nu_f((1+\varepsilon)r)+p}^{\infty}|f_k|(r(1+\varepsilon))^k(1+\varepsilon)^{-k}\le$$

$$\le\mu_f(r)\left(\nu_f((1+\varepsilon)r)+p\right)+\mu_f((1+\varepsilon)r)(1+\varepsilon)^{-\nu_f((1+\varepsilon)r)-p+1}/\varepsilon\,.$$

But

$$\ln\mu_f((1+\varepsilon)r)=\ln\mu_f(r)+\int_r^{(1+\varepsilon)r}\frac{\nu_f(t)}{t}dt\le\nu_f((1+\varepsilon)r)\ln(1+\varepsilon)\,,$$

and we get the required inequality. Theorem 5 is proved. ►

2.3. Newton's majorant. Let's find another entire function $\tilde{f}$, which has the same maximum term as f [1]. To do this, we will construct points $A_n\left(n;-\ln|f_n|\right)$ on the plane. Starting from a point $A_{n_1}=A_0$ (assume that $f_0\ne 0$), we draw a vertical ray l_1 downward and rotate it counterclockwise around the point A_{n_1} until it intersects one of the points $A_n\ne A_{n_1}$. Let the ray l_1 coincide with the ray $\tilde{l}_1$ at the time of intersection. Along this ray $\tilde{l}_1$, besides A_{n_1} there is at least one more point A_n. The farthest one of these points from A_{n_1} (there are finitely many such points, because $\frac{1}{n}\ln\frac{1}{|a_n|}\to+\infty$) is denoted as A_{n_2}. Next, from A_{n_2} we draw a vertical ray l_2 downward again and repeat the process, rotating it around the point A_{n_2}. This results in an infinite number of points A_{n_k}. Connecting these points successively by segments, we obtain a convex polygon G, which is called the Newton polygon. We then project the points A_n onto the polygon G, yielding points $\tilde{A}_n(n;-\ln \tilde{f}_n)$. It is clear that $\tilde{f}_n = \exp(-G(n))$, where the $y = G(x)$ is the equation of the polygon G. An entire function $\tilde{f}$ with Taylor coefficients $\tilde{f}_n$ is called the Newton majorant of the function f. From the construction, it follows that $\tilde{f}_{n_k} = |f_{n_k}|$ and $|f_n| \le \tilde{f}_n$ for all n. In this case, $\ln \kappa_n(\tilde{f})$ is the angular coefficient of the segment joining the points $\tilde{A}_{n-1}$ and $\tilde{A}_n$, where $\chi_n(f) = |f_{n-1}/f_n|$. Since G is a convex polygon, the sequence $(\chi_n(\tilde{f}))$ is non-decreasing. Moreover, if

$$0 < \chi_1(f) \le \chi_2(f) \le ..., \tag{1}$$

then $|f_n| = \tilde{f}_n$. Since f is an entire function, we have [1]

$$\lim_{n\to\infty} \kappa_n(\tilde{f}) = \lim_{n\to\infty}\left(\sqrt[n]{\tilde{f}_n}\right)^{-1} = \lim_{n\to\infty}\left(\sqrt[n_k]{|f_{n_k}|}\right)^{-1} = +\infty$$

Hence, $\tilde{f}$ is an entire function. The points A_{n_k} are called the vertices of the Newton polygon.

Remark 1. *If* $f_0 = f_1 = ... f_{s-2} = 0$, $f_{s-1} \ne 0$, *then by the definition* $\tilde{f}_0 = \tilde{f}_1 = ... \tilde{f}_{s-2} = 0$ *and the construction of the Newton polygon, we start from the point* $A_{n_1} = A_{s-1}$, *drawing a vertical ray* l_1 *downward from it. In this case, the Newton polygon also includes the ray* $\tilde{l}_0$ *parallel to the ordinate axis, starting from the point* $A_{n_1} = A_{s-1}$. *Then* $0 < \chi_s(f) \le \chi_{s+1}(f) \le ...$. *It can also be stated that in the considered case, the Newton majorant of the function* f *is the function* $\tilde{f}(z) = z^{s-1}\tilde{F}(z)$, *where* $\tilde{F}$ *is the Newton majorant of the function* $F(z) = f_{s-1} + f_s z + ...$. *Moreover,* $\mu_F(r) = r^s \mu_f(r)$ $\mu_F(r) = r^s \mu_f(r)$.

The main properties of the Newton majorant can be summarized in the form of the following statements.

Theorem 1 ([1]). *For every entire transcendental function* f *holds* $|f_k| \le \tilde{f}_k$ *and* $\tilde{f}_{n_k} = |f_{n_k}|$ *for all* $k \in \mathbb{Z}$.

Theorem 2 ([1]). *For every entire function* f *holds* $M_f(r) \le M_{\tilde{f}}(r)$ *for all* $r \in [0;+\infty)$.

Theorem 3 ([1]). *For every entire function* f *holds*

$$0<\chi_s(\tilde f)\le \chi_{s+1}(\tilde f)\le ...\le \chi_k(f)\to\infty,$$

for all $k\in\mathbb{Z}$, *where* s *is the smallest natural number for which* $f_{s-1}\ne 0$.

Theorem 4 ([1]). *For every entire transcendental function* f *and all* $k\in\mathbb{Z}$ *holds* $\mu_{\tilde f}(r)=\left|\tilde f_{s-1}\right|r^{s-1}$ *if* $r\in[0;\chi_s(\tilde f)]$, *and* $\mu_{\tilde f}(r)=\left|\tilde f_m\right|r^m$ *if* $r\in[\chi_m(\tilde f);\chi_{m+1}(\tilde f)]$ *and* $m\ge s$, *where* s *is the smallest natural number for which* $f_{s-1}\ne 0$.

Theorem 5 ([1]). *For every entire transcendental function* f *holds*

$$\chi_{n_k+1}(\tilde f)=\chi_{n_k+2}(\tilde f)=...=\chi_{n_{k+1}}(\tilde f)=$$
$$=\left|\tilde f_{n_k}/\tilde f_{n_{k+1}}\right|^{1/(n_{k+1}-n_k)}=\left|f_{n_k}/f_{n_{k+1}}\right|^{1/(n_{k+1}-n_k)}$$

for all $k\in\mathbb{N}$. *In this case, the sequence*

$$\rho_k=\left|\tilde f_{n_k}/\tilde f_{n_{k+1}}\right|^{1/(n_{k+1}-n_k)},\ \rho_0:=0,$$

is increasing, $\rho_k\to\infty$, *and*

$$\mu_{\tilde f}(\rho_{k-1})=\left|\tilde f_{n_k}\right|(\rho_{k-1})^{n_k}=\left|\tilde f_{n_{k-1}}\right|(\rho_{k-1})^{n_{k-1}}=$$
$$=\left|\tilde f_{n_k}\right|\left(\chi_{n_k}\right)^{n_k}=\left|\tilde f_{n_k+1}\right|\left(\chi_{n_k+1}\right)^{n_k+1}...=\left|\tilde f_{n_{k+1}-1}\right|\left(\chi_{n_{k+1}-1}\right)^{n_{k+1}-1}.$$

Theorem 6 ([1]). *For every entire transcendental function* f *the function* $\nu_{\tilde f}$ *is non-decreasing and continuous on the right, the set* $\{n_k:k\in\mathbb{N}\}$ *is the set of its values, the points* ρ_k *are its points of discontinuity and for all* $k\in\mathbb{N}$ *holds* $\mu_{\tilde f}(r)=\left|\tilde f_{n_k}\right|r^{n_k}$ *and* $\nu_{\tilde f}(r)=n_k$ *for* $r\in[\rho_{k-1};\rho_k)$.

Theorem 7 ([1]). *For every entire transcendental function* f *and all* $r\in[0;+\infty)$ *holds* $\mu_{\tilde f}(r)=\mu_f(r)$ *and* $\nu_f(r)=\nu_{\tilde f}(r)$.

Theorem 8 ([1]). *If* f *is an entire function and all* $f_n\ne 0$, *then* $\nu_f(r)=\max\{k:\chi_k(\tilde f)\le r\}=-1+\sum_{\kappa_k(\tilde f)\le r}1$, *where* $\chi_0(\tilde f):=0$.

Theorem 9 ([1]). *For every entire transcendental function* f *and all* $k_0\in\mathbb{Z}_+$ *there exists* $r_0\in\mathbb{R}$ *such that* $\mu_f(r)=\max\left\{\left|f_n\right|r^n:n\ge k_0\right\}$ *for all* $r\in[r_0;+\infty)$.

2.4. Φ **-type of an entire function.** Let f and Φ be entire transcendental functions, and let Ψ_Φ^{-1} be the function inverse to

$\Psi_\Phi(x) = \ln M_\Phi(x)$. The Φ-type of an entire function f is defined as the number $q = q_\Phi^f$, determined by the equality [1]:

$$q = \overline{\lim_{r\to+\infty}} \frac{\Psi_\Phi^{-1}\left(\ln^+ M_f(r)\right)}{r}.$$

The Φ-type of an entire function f can also be defined as the exact infimum of those $q_1 \in [0;+\infty]$ for which $(\exists r_0 \in \mathbb{R})(\forall r \ge r_0): M_f(r) \le M_\Phi(q_1 r)$. In this definition, the maximum of the modulus can be replaced by the maximum term. That is, the Φ-type of an entire function f is the exact infimum of those $q_1 \in [0;+\infty]$ for which $(\exists r_0 \in \mathbb{R})(\forall r \ge r_0): \mu_f(r) \le \mu_\Phi(q_1 r)$.

Theorem 1 ([1]). *If the sequence* $\chi_n(\Phi) = \left|\Phi_{n-1}/\Phi_n\right|$ *is non-decreasing, then*

$$q_\Phi^f = \overline{\lim_{n\to\infty}} \sqrt[n]{|f_n/\Phi_n|}. \qquad (1)$$

Proof. Indeed, let us denote the right-hand side of (1) by η_Φ^f. Then

$$(\forall \varepsilon > 0)(\exists c_1)(\forall n): |f_n| \le c_1 |\Phi_n| \eta^n,\ \eta := \eta_\Phi^f + \varepsilon.$$

Therefore, $\mu_f(r) \le c_1 \mu_f(\eta r)$, and hence, $q_\Phi^f \le \eta_\Phi^f$. Now, let's prove the opposite inequality. If $q_\Phi^f = +\infty$, then it is obvious. Let $q_\Phi^f < +\infty$. Then

$$(\forall \varepsilon > 0)(\exists r_0)(\forall r \ge r_0): \mu_f(r) \le \mu_\Phi(qr),\ q := q_\Phi^f + \varepsilon.$$

Thus, $|f_n| r^n \le \mu_\Phi(qr),\ n \ge 0,\ r \ge r_0$. Taking $r = \chi_n(\Phi)/q$ in the last inequality, we get $|f_n|(\chi_n(\Phi)/q)^n \le \mu_\Phi(\chi_n(\Phi)) = |\Phi_n| \chi_n^n(f)$, that is $\eta_\Phi^f \le q_\Phi^f$. The theorem is proved. ►

Example 1. *If* $\Phi(z) = e^z$, *then* $\Psi_\Phi(x) = x$, $\Psi_\Phi^{-1}(x) = x$, $q_\Phi^f = \overline{\lim_{r\to+\infty}} \frac{\ln^+ M_f(r)}{r}$ *and* $q_\Phi^f = \overline{\lim_{k\to\infty}} \sqrt[k]{|k!\, f_k|} = \overline{\lim_{k\to\infty}} \frac{k}{e} \sqrt[k]{|f_k|}$.

2.5. The Poisson formula and the Schwarz formula. The Schwarz-Jensen formula and the Poisson-Jensen formula. Let $U(a;R) = \{z : |z-a| < R\}$ and f is a function holomorphic in a closed disk $\overline{U(0;R)} := \{z : |z| \le R\}$, where $0 < R < +\infty$.

Theorem 1 ([24, 28, 31, 34, 45, 47, 51, 54]). *Let* $0 < R < +\infty$ *and* $U(a;R) = \{z : |z-a| < R\}$. *If the function* f *is holomorphic in* $\overline{U(0;R)}$, *then the following Poisson formula*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\theta})\frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2}\,d\theta, \quad |z| < R, \tag{1}$$

and the Schwarz formula

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re} f(Re^{i\theta})\frac{Re^{i\theta} + z}{Re^{i\theta} - z}\,d\theta + i\operatorname{Im} f(0), \tag{2}$$

hold for $z \in U(0;R)$.

Proof. By Cauchy's integral formula, we have

$$\frac{1}{2\pi i}\int_{\partial U(0;R)} \frac{f(t)}{t-z}\,dt = \begin{cases} f(z), & z \in U(0;R), \\ 0, & z \notin \overline{U(0;R)}. \end{cases}$$

This yields (1) and (2) for $z = 0$. Let $z \in U(0;R)$, $z \neq 0$ and $z^{\bullet} = R^2/\overline{z}$. Then

$$\frac{1}{2\pi i}\int_{\partial U(0;R)} \frac{f(t)}{t-z^{\bullet}}\,dt = 0.$$

Therefore, for $z \in U(0;R)$, we get

$$f(z) = \frac{1}{2\pi i}\int_{\partial U(0;R)} f(t)\left(\frac{1}{t-z} - \frac{1}{t-z^{\bullet}}\right)dt = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\theta})\frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2}\,d\theta.$$

Thus, the formula (1) is proved. If $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$, then from (1) we obtain

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta})\frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2}\,d\theta, \quad |z| < R, \tag{3}$$

$$1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2}\,d\theta, \quad |z| < R, \tag{4}$$

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta})\,d\theta. \tag{5}$$

It is easy to see that (here $z = re^{i\varphi}$)

$$\operatorname{Re}\frac{Re^{i\theta}+z}{Re^{i\theta}-z}=\frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}=\frac{R^2-r^2}{R^2-2Rr\cos(\varphi-\theta)+r^2}.$$

Let's consider the function

$$f_1(z)=\frac{1}{2\pi}\int_0^{2\pi}u(Re^{i\theta})\frac{Re^{i\theta}+z}{Re^{i\theta}-z}d\theta=\frac{1}{2\pi R}\int_{\partial U(0;R)}u(t)\frac{t+z}{t-z}|dt|.$$

This function f_1 is holomorphic in $U(0;R)$ and $\operatorname{Re} f_1=u=\operatorname{Re} f$. Therefore, $f=f_1+\text{const}$. Indeed, if $u_1=\operatorname{Re} f_1$ and $v_1=\operatorname{Im} f_1$, then from the Cauchy-Riemann conditions, we find

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=\frac{\partial u_1}{\partial x}=\frac{\partial v_1}{\partial y},$$

that is

$$\frac{\partial v}{\partial y}=\frac{\partial v_1}{\partial y}.$$

This implies $v(x,y)=v_1(x,y)+c_1(x)$, where $c_1(x)$ does not depend on y. Moreover, from the Cauchy-Riemann conditions, we obtain

$$\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}=-\frac{\partial u_1}{\partial y}=-\frac{\partial v_1}{\partial x},\ \frac{\partial v}{\partial x}=\frac{\partial v_1}{\partial x},$$

and therefore, $v(x,y)=v_1(x,y)+c_2(y)$. Hence, the functions $c_1(x)$ and $c_2(y)$ are constant $(=c_0)$. Further, $v=v_1+c_0$, $f=f_1+c_0$ and

$$f(z)=\frac{1}{2\pi}\int_0^{2\pi}u(Re^{i\theta})\frac{Re^{i\theta}+z}{Re^{i\theta}-z}d\theta+ic_0.$$

Furthermore, using (5), we obtain

$$u(0)+iv(0)=\frac{1}{2\pi}\int_0^{2\pi}u(R\mathrm{e}^{i\theta})d\theta+c_0=u(0)+c_0,\ c_0=v(0).$$

From this and the previous equality it follows (2). The theorem is proved. ►

Corollary 1. *If the function u is harmonic in $\overline{U(0;R)}$, then the Poisson formula (3) holds.*

Proof. Indeed, a function is called harmonic in the closed disk $\overline{U(0;R)}$ if it is harmonic in some domain $G\subset\overline{U(0;R)}$. This domain G can be considered simply connected. Then, there exists a function f holomorphic in the domain G such that $\operatorname{Re} f=u$. From here and from the proof above it follows (3). Corollary 1 is proved. ►

Theorem 2 ([24, 28, 31, 34, 45, 47, 51, 54]). *If the function f is holomorphic in $\overline{U(0;R)}$ and has no zeros there, then the Schwarz-Jensen formula*

$$\ln f(z) = \frac{1}{2\pi}\int_0^{2\pi} \ln\left| f(Re^{i\theta})\right| \frac{Re^{i\theta}+z}{Re^{i\theta}-z}\, d\theta + ic_0 \qquad (6)$$

and the Poisson-Jensen formula

$$\ln\left| f(z)\right| = \frac{1}{2\pi}\int_0^{2\pi} \ln\left| f(Re^{i\theta})\right| \frac{R^2-|z|^2}{\left|Re^{i\theta}-z\right|^2}\, d\theta\,, \qquad (7)$$

are valid, where $\ln f(z)$ is an arbitrary holomorphic branch of the function $\operatorname{Ln} f(z)$ in $\overline{U(0;R)}$ and $c_0 = \operatorname{Im}\ln f(0)$.

Proof. To derive these formulas, it is sufficient to apply formulas (1) and (2) to the function $\ln f(z)$. The theorem is proved. ►

2.6. Zeros of holomorphic functions. The Nevanlinna-Jensen formula. Jensen's equality. The zero of a function f is a number a for which $f(a)=0$. In other words, a zero of a function is a number that is a solution to the equation $f(z)=0$. Let the function f be holomorphic in a domain D. A zero $a\in D$ of the function f is called a zero of finite order $m\in\mathbb{N}$ or a zero of multiplicity $m\in\mathbb{N}$ if [3, 26, 34])

$$f(a) = f'(a) = \ldots = f^{(m-1)}(a) = 0,\;\; f^{(m)}(a)\neq 0\,.$$

A zero $a\in D$ is called a zero of infinite order if $(\forall n\in\mathbb{Z}_+): f^{(n)}(a)=0$. A zero of order $m=1$ is called a simple zero. In complex analysis, it is proved that all zeros of a holomorphic function $f\neq 0$ in a domain D have finite multiplicity. Furthermore, the following statement is proved.

Theorem 1 ([3, 26, 34]). *Let the function f be holomorphic in a domain D. Then the following conditions are equivalent:*

1) the function f has a zero of order $m\in\mathbb{N}$ at the point $a\in D\setminus\{\infty\}$;

2) the function f can be represented in the form $f(z)=(z-a)^m g(z)$, where g is holomorphic function at the point a and $g(a)\neq 0$;

3) the Taylor series expansion of the function f in the neighborhood of a has the form $f(z)=\sum_{k=m}^{\infty} b_k (z-a)^k$.

Theorem 1 is proved in complex analysis.

Let $Z=\{z_k\}$ be the set of zeros of a function f holomorphic in a domain D, and let m_k denote the order (multiplicity) of a zero z_k. The set of all ordered pairs $\{(z_k;m_k)\}$ is called [3, 26, 34] the divisor of zeros of the function f. The sequence of zeros of the function f is the sequence (λ_n) constructed as follows:

$$\lambda_1=\lambda_2=...=\lambda_{m_1}=z_1,\ \lambda_{m_1+1}=\lambda_{m_1+2}=...=\lambda_{m_2}=z_2,\dots.$$

The number of zeros of the function f on a set $E\subset D$ is defined as the number $n_{f,E}=\sum_{z_k\in E}m_k$, i.e, $n_{f,D}=\sum_{\lambda_k\in E}1$. According to the uniqueness theorem (also proved in complex analysis), if the function $f\neq 0$ is holomorphic in a domain D, then it has a finite number of zeros (or none at all) on any compact from D [3, 26, 34].

Example 1. *If* $f(z)=z^2(z-1)\sin\pi z$, *then the set* $Z=\{0;1;-1;2;-2;...\}$ *is the set of zeros of* f *and the sequence* $\lambda=(0;0;0;1;1;-1;2;-2;3;-3;...)$ *is a sequence of its zeros.*

Theorem 2 ([24, 28, 31, 34, 45, 47, 51, 54]). *Let* $0<R<+\infty$, *the function* $f\neq 0$ *is holomorphic in* $\overline{U(0;R)}:=\{z:|z|\leq R\}$, *has a zero of multiplicity* m *at the point* 0 *and* (λ_n) *be a sequence of its zeros. Then the Nevanlinna-Jensen formula*

$$\ln|f(z)|=\frac{1}{2\pi}\int_0^{2\pi}\ln|f(Re^{i\theta})|\operatorname{Re}\frac{Re^{i\theta}+z}{Re^{i\theta}-z}d\theta+\sum_{0<|\lambda_k|\leq R}\ln\left|\frac{R(z-\lambda_k)}{R^2-z\overline{\lambda_k}}\right|+m\ln\frac{|z|}{R} \qquad (1)$$

is valid for $z\in U(0;R)$.

Proof. Indeed, to obtain (1), it is enough to apply formula (4) from section 2.5 to the function

$$\Psi(z)=\frac{f(z)}{z^m\prod\limits_{0<|\lambda_K|\leq R}\frac{R(z-\lambda_k)}{R^2-z\overline{\lambda_k}}},$$

which satisfies all the conditions of Theorem 2 in Section 2.5, and taking into account that (the denominator is conjugate to the numerator)

$$\left|\frac{R(Re^{i\theta}-\lambda_k)}{R^2-Re^{i\theta}\overline{\lambda_k}}\right|=\left|\frac{Re^{i\theta}-\lambda_k}{Re^{-i\theta}-\overline{\lambda_k}}\right|=1,$$

and (see the formula (4) from section 2.5)

$$\frac{1}{2\pi}\int_0^{2\pi} m\ln|z|\operatorname{Re}\frac{Re^{i\theta}+z}{Re^{i\theta}-z}d\theta = m\ln|z|.$$

Theorem 2 is proved. ►

Theorem 3 ([24, 28, 31, 34, 45, 47, 51, 54]). *Let* $0<R<+\infty$, *the function* $f\neq 0$ *is holomorphic in* $\overline{U(0;R)}:=\{z:|z|\le R\}$, $f(0)=1$ *and* (λ_n) *be a sequence of its zeros. Then the Jensen equality holds*

$$\sum_{0<|\lambda_k|\le R}\ln\frac{R}{|\lambda_k|}=\frac{1}{2\pi}\int_0^{2\pi}\ln\left|f(Re^{i\theta})\right|d\theta. \qquad (2)$$

Proof. To obtain formula (2), it is enough to substitute $z=0$ in the Nevanlinna-Jensen formula

$$\ln|f(z)|=\frac{1}{2\pi}\int_0^{2\pi}\ln\left|f(Re^{i\theta})\right|\operatorname{Re}\frac{Re^{i\theta}+z}{Re^{i\theta}-z}d\theta+\sum_{0<|\lambda_k|\le R}\ln\left|\frac{R(z-\lambda_k)}{R^2-z\overline{\lambda_k}}\right|.$$

Theorem 3 is proved. ►

Theorem 4 ([24, 28, 31, 34, 45, 47, 51, 54]). *Let* $0<R<+\infty$, *the function* $f\neq 0$ *is holomorphic in* $\overline{U(0;R)}$, *has a zero of multiplicity* m *at the point* 0 *and* (λ_n) *be a sequence of its zeros. Then the Jensen equality holds*

$$\sum_{0<|\lambda_k|\le R}\ln\frac{R}{|\lambda_k|}+m\ln R=\frac{1}{2\pi}\int_0^{2\pi}\ln\left|f(Re^{i\theta})\right|d\theta-\ln\left|\frac{f^{(m)}(0)}{m!}\right|. \qquad (3)$$

Proof. To obtain formula (3), it is enough to passing to the limit as $z\to 0$ in the Nevanlinna-Jensen formula or apply formula (2) to the function $F(z)=m!\,f(z)/f^{(m)}(0)$. The theorem is proved. ►

Corollary 1. *Let* $0<R<+\infty$ *and the function* $f\neq 0$ *is holomorphic in* $\overline{U(0;R)}$. *Then the Jensen inequality holds*

$$N(r)\le\ln M_f(r)+c_1,\ r\in[0;R],$$

where the constant c_1 *depends only on* f. *In this case,* $c_1=0$ *if* $f(0)=1$.

2.7. Upper bounds for a holomorphic function via modulus of its real part. Lower bounds for holomorphic functions

Theorem 1 ([24, 28, 31, 34, 45, 47, 51, 54]). *For every function* f *holomorphic in* $\overline{U(0;R)}$, $0<R<+\infty$, *holds*

$$M_f(r)\le|f(0)|+(A_f(R)-\operatorname{Re}f(0))2r/(R-r),$$

where $0<r<R$ *and* $A_f(r)=\max\{\operatorname{Re}f(z):|z|=r\}$.

Proof. From the Schwartz formula, taking into account that

$$u(0)=\frac{1}{2\pi}\int_0^{2\pi}u(Re^{i\theta})d\theta,$$

we obtain

$$f(z)=\frac{1}{\pi}\int_0^{2\pi}u(Re^{i\theta})\frac{z}{Re^{i\theta}-z}d\theta+f(0),\ |z|<R,$$

$$0=\frac{1}{\pi}\int_0^{2\pi}\frac{z}{Re^{i\theta}-z}d\theta.$$

Therefore,

$$f(z)=-\frac{1}{\pi}\int_0^{2\pi}(A_f(R)-u(Re^{i\theta}))\frac{z}{Re^{i\theta}-z}d\theta+f(0).$$

Since $A_f(R)-u(Re^{i\theta})\ge 0$, we have

$$|f(z)|\le\frac{1}{\pi}\frac{r}{R-r}\int_0^{2\pi}(A_f(R)-u(Re^{i\theta}))d\theta+|f(0)|=$$

$$=|f(0)|+(A_f(R)-u(0))2r/(R-r),$$

whence it follows the required proposition. Theorem 1 is proved. ►

Corollary 1. *If* $0<R<+\infty$, *the function* f *is holomorphic in* $\overline{U(0;R)}$, $f(0)=1$ *and* f *has no zeros in* $\overline{U(0;R)}$, *then*

$$\ln|f(z)|\ge-\frac{2r}{R-r}\ln M_f(R),\ r=|z|<R.$$

Theorem 2 ([24, 28, 31, 34, 45, 47, 51, 54]). *If* $0<R<+\infty$, *the function* f *is holomorphic in* $\overline{U(0;2R)}$, $f(0)=1$ *and* f *has no zeros in* $\overline{U(0;2R)}$, *then*

$$\ln|f(z)|\ge-\frac{2r}{R-r}\ln M_f(R),\ r=|z|<R.$$

Proof. Indeed, let's consider any holomorphic branch ψ of the function $\operatorname{Ln} f(z)$. Then $\operatorname{Re}\psi(z)=\ln|f(z)|$, $\psi(0)=0$, and by Corollary 1, we obtain

$$-\ln|f(z)|\le\big|\ln|f(z)|\big|\le|\psi(z)|\le\max\{\ln|f(z)|:|z|\le r\}2r/(R-r)=$$

$$=\ln M_f(r)2r/(R-r),$$

whence it follows the required proposition. Theorem 2 is proved. ►

Theorem 3 (Cartan) ([24, 28, 31, 45, 47, 51, 54]). *Let* (λ_k), $k\in\overline{1;n}$, *be a finite sequence of points* $\lambda_k\in\mathbb{C}$ *and* $h>0$ *be a given number. Then, in*

$\mathbb{C}$ *there exists a system* Ω *of disks with the sum of the radii equal to* $2h$, *such that*

$$(\forall z \notin \Omega): \left| \prod_{k=1}^{n} (z - \lambda_k) \right| > (h/e)^n.$$

Theorem 4 ([24, 28, 31, 45, 47, 51, 54]). *Let* $0 < R < +\infty$, *the function* f *is holomorphic in* $\overline{U(0;2eR)}$, $f(0) = 1$ *and* $0 < \eta < 3e/2$. *Then inside the disk* $\overline{U(0;R)}$, *but outside of a family of excluded disks the sum of whose radii is not greater than* $4\eta R$, *we have*

$$\ln|f(z)| \geq -H(\eta) \ln M_f(2eR), \quad H(\eta) = 2 + \ln \frac{3e}{2\eta}.$$

Proof. Let $n = n(2R)$ be the number of zeros of the function f in the disk $\overline{U(0;2R)}$, and

$$g(z) = \frac{(-2R)^n}{\prod_{|\lambda_k| \leq 2R} \lambda_k} \prod_{|\lambda_k| \leq 2R} \frac{2R(z - \lambda_k)}{(2R)^2 - \overline{\lambda_k} z},$$

where λ_k, $k \in \overline{1,n}$, are the zerof of f in $\overline{U(0;2R)}$. Then the function $F = f/g$ is holomorphic in $\overline{U(0;2R)}$, has no zeros in $\overline{U(0;2R)}$, $F(0) = 1$, and

$$M_F(2R) = M_f(2R) \prod_{|\lambda_k| \leq 2R} (|\lambda_k|/2R),$$

$$\ln M_F(2R) = \ln M_f(2R) - N(2R) \leq \ln M_f(2R).$$

Therefore, by Theorem 3

$$\ln|f(z)| \geq -\frac{2R}{2R - R} 2 \ln M_f(2R) = -2 \ln M_f(2eR), \ |z| \leq R.$$

But, for $z \in \overline{U(0;R)}$

$$\left| \prod_{|\lambda_k| \leq 2R} ((2R)^2 - \overline{\lambda_k} z) \right| \leq (6R^2)^n.$$

According to Theorem 3, outside of a family of excluded disks with the sum of the radii equal to $4\eta R$, we have

$$\left| \prod_{|\lambda_k| \leq 2R} 2R(z - \lambda_k) \right| \geq (2R)^n \left(2\eta R/e\right)$$

and consequently

$$\ln|g(z)| \geq N(2R) + \ln \frac{(2R)^n (2\eta R)^n}{e^n (R^2)^n} \geq n \ln \frac{2\eta}{3e}, \ |z| \leq R.$$

From the Jensen inequality, we obtain

$$n = n(2R) \leq \int_{2R}^{2eR} \frac{n(2R)}{t} dt \leq \int_0^{2eR} \frac{n(t)}{t} dt = N(2eR) \leq \ln M_f(2eR).$$

Hence,

$$\ln|g(z)| \geq \ln M_f(2eR) \ln \frac{2\eta}{3e},$$

$$\ln|f(z)| \geq \ln|g(z)| - 2\ln M_f(2eR) \geq -2\ln M_f(2eR) + \ln \frac{2\eta}{3e} \ln M_f(2eR),$$

and we get the required inequality. The theorem is proved. ►

Corollary 2 ([24, 28, 31, 45, 47, 51, 54]). *Let f be an entire transcendental function. Then there exist a number $c_1 > 1$ and a sequence (r_k), $0 < r_k \uparrow +\infty$, $r_{k+1}/r_k = O(1)$ as $k \to +\infty$, such that*

$$\varliminf_{k\to\infty} \frac{\ln \min\{\ |f(z)| : |z| = r_k\}}{\ln M_f(c_1 r_k)} > -\infty.$$

Proof. Indeed, let $R_k = 2^k$. Put $R = R_k$ and $\eta = 1/16$ as in Theorem 4. Then, in the disk $U(0; R_k)$, outside a system of exceptional disks with the sum of the radii equal to $4\eta R_k$, we have

$$\ln|f(z)| \geq -H(\eta) \ln M_f(2eR_k)$$

Since $4R_k\eta = R_k/4 < 2^{k-1} = R_k - R_{k-1}$, there exists a circle $\partial U(0; r_k)$, $R_{k-1} < r_k < R_k$, which does not intersect the exceptional disks. It remains to note that $R_k \leq 2r_k$ and, therefore, $r_{k+1}/r_k \leq 4$. ►

2.8. Counting functions of sequences. Let (λ_k) be a sequence of complex numbers such that $0 \leq |\lambda_1| \leq |\lambda_2| \leq \ldots$, and let $n(t)$ denote the number of terms of a sequence (λ_k) for which $|\lambda_k| \leq t$, i.e., $n(t) = \sum_{|\lambda_k| \leq t} 1 = \max\{k : |\lambda_k| \leq t\}$. The function $n(t)$ is called the counting function of the sequence (λ_k), and the function [24, 28, 31, 33, 45, 47, 51, 52, 54]

$$N(r) = \int_0^r \frac{n(t) - n(0)}{t} dt + n(0) \ln r, \ r > 0,$$

is called the Nevanlinna counting function or the averaged counting function of the sequence (λ_k). Since

$$\sum_{0<|\lambda_k|\le r} \ln\frac{r}{|\lambda_k|} = \int_0^r \ln\frac{r}{t} d\left(n(t)-n(0)\right) = \int_0^r \frac{n(t)-n(0)}{t}dt,$$

we have

$$N(r) = \sum_{0<|\lambda_k|\le r} \ln\frac{r}{|\lambda_k|} + n(0)\ln r.$$

If all $\lambda_k \ne 0$, then

$$N(r) = \sum_{|\lambda_k|\le r} \ln\frac{r}{|\lambda_k|} = \ln\frac{r^{n(r)}}{\prod_{|\lambda_k|\le r}|\lambda_k|} = \max_{k\ge 0}\frac{r^k}{\prod_{n=1}^{k}|\lambda_n|}.$$

Example 1. *Let* $\lambda_k = 2k/3$. *Then for a given* $t \ge \lambda_1$ *there is* m *such that* $\lambda_m \le t < \lambda_{m+1}$, $n(t) = m = 3\lambda_m/2 \le 3t/2$ *and*

$$n(t) = m+1-1 = 3\lambda_{m+1}/2 - 1 > 3t/2 - 1.$$

Hence, we have $n(t) = 3t/2 + O(1)$ *as* $t \to +\infty$, *and* $N(r) = 3r/2 + O(\ln r)$ *as* $r \to +\infty$.

Example 2. *Let* $\lambda_k = q^{k-1}$, $|q|>1$. *Then* $n(t) = \dfrac{\log t}{\log|q|} + O(1)$ *as* $t \to +\infty$ *and*

$$N(r) = \frac{\ln^2 r}{2\ln|q|} + \frac{\ln r}{2} + O(1),\ r \to +\infty.$$

Indeed, let $t>1$. *Then there is* m *such that* $|q|^{m-1} \le t < |q|^m$. *Therefore,* $n(t) = m$, $m > \dfrac{\log t}{\log|q|}$, $m < \dfrac{\log t}{\log|q|} + 1$ *and* $n(t) = \dfrac{\log t}{\log|q|} + O(1)$ *as* $t \to +\infty$. *Further, if* $r>1$ *then there exists* m *such that* $|q|^{m-1} \le r < |q|^m$. *Furthermore,* $r = \alpha|q|^{m-1}$, $1 \le \alpha < |q|$. *Thus*

$$N(r) = \sum_{k=1}^{m}\ln\frac{r}{|q|^{k-1}} = m\ln r - \ln|q|\sum_{k=0}^{m-1}k =$$

$$= m\left(\ln r - \frac{m-1}{2}\ln|q|\right) = \frac{\ln r - \ln\alpha + \ln|q|}{2\ln|q|}(\ln r + \ln\alpha) =$$

$$= \frac{\ln^2 r}{2\ln|q|} - \frac{\ln^2 \alpha}{2\ln|q|} + \frac{\ln r}{2} + \frac{\ln|q|\ln\alpha}{2\ln|q|} = \frac{\ln^2 r}{2\ln|q|} + \frac{\ln r}{2} + O(1),\ r \to +\infty.$$

***Example 3* ([1]).** *Let* $q \in [0;+\infty)$, *let* $(\lambda_{k,1})$ *and* $(\lambda_{k,2})$ *be two sequences of nonzero complex numbers, each of which has a single accumulation point at* ∞, *and let* $N_1(r)$ *and* $N_2(r)$ *denote their averaged counting functions. Then the following conditions*

$$N_1(r) \le N_2((q+o(1))r),\ r \to +\infty,$$

and

$$\overline{\lim_{n\to\infty}} \prod_{k=1}^{n} |\lambda_{k,2}/\lambda_{k,1}|^{1/n} \le q$$

are equivalent. The conditions

$$N_1(r) = N_2((q+o(1))r),\ r \to +\infty,$$

and

$$\lim_{n\to\infty} \prod_{k=1}^{n} |\lambda_{k,2}/\lambda_{k,1}|^{1/n} = q$$

are also equivalent.

2.9. Zeros of entire functions. It is known that every polynomial $f(z) = \sum_{k=0}^{n} f_k z^k$ of degree n has exactly n zeros λ_k (counted with multiplicities) in $\mathbb{C}$ and can be represented in the form [24, 28, 31, 33, 45, 47, 51, 54]:

$$f(z) = f_n \prod_{k=1}^{n} (z - \lambda_k) = cz^m \prod_{\lambda_k \ne 0} (1 - z/\lambda_k),$$

where m is the multiplicity of the zero $z = 0$, and c is some constant. Now, let's consider similar expansions for entire transcendental functions, which can be regarded as polynomials of infinite degree. An entire transcendental function either has no zeros, or it has a finite number of zeros, or it has an infinite number of zeros. If the number of zeros of an entire function f is infinite, and (λ_k) is the sequence of its zeros, then $\lim_{k\to\infty} \lambda_k = \infty$. Moreover, the following statement is true.

Theorem 1 ([24, 28, 31, 33, 45, 47, 51, 54]). *If the sequence* (λ_k) *is a subsequence of the zeros of an entire function* $f \ne 0$, *then Jensen's inequality holds*

$$N(r) \le \ln M_f(r) + c_1,\ r \in [0;+\infty), \qquad (1)$$

where c_1 *is a constant,* $n(t)=\sum_{|\lambda_k|\le t}1$, $N(r)=\int_0^r \frac{n(t)-n(0)}{t}dt+n(0)\ln r$ *and* $M_f(r)=\max\left\{\left|f(re^{i\theta})\right|:\theta\in[0;2\pi]\right\}$.

Proof. Indeed, according to Jensen's equality

$$\sum_{0<|z_k|\le r}\ln\frac{r}{|z_k|}+m\ln r=\frac{1}{2\pi}\int_0^{2\pi}\ln\left|f(re^{i\theta})\right|d\theta-\ln\left|\frac{f^{(m)}(0)}{m!}\right|,$$

where (z_k) are the zeros of a function and m is the multiplicity of the zero 0. Moreover, $n(0)\le m$ and $\sum_{0<|\lambda_k|\le r}\ln\frac{r}{|\lambda_k|}\le\sum_{0<|z_k|\le r}\ln\frac{r}{|z_k|}$. Therefore,

$$\sum_{0<|\lambda_k|\le r}\ln\frac{r}{|\lambda_k|}+n(0)\ln r\le\frac{1}{2\pi}\int_0^{2\pi}\ln\left|f(re^{i\theta})\right|d\theta-\ln\left|\frac{f^{(m)}(0)}{m!}\right|.$$

In addition,

$$\frac{1}{2\pi}\int_0^{2\pi}\ln\left|f(re^{i\theta})\right|d\theta\le\ln M_f(r)$$

and

$$\sum_{0<|\lambda_k|\le r}\ln\frac{r}{|\lambda_k|}+n(0)\ln r=\int_0^r\ln\frac{r}{t}d\left(n(t)-n(0)\right)+n(0)\ln r=$$

$$=\int_0^r\frac{n(t)-n(0)}{t}dt+n(0)\ln r=N(r).$$

Therefore, we obtain the inequality (1). Theorem 1 is proved. ►

Remark 1. *The inequality (1) indicates that an entire function cannot have too many zeros. Entire transcendental functions, metaphorically speaking, are polynomials of infinite degree. Therefore, one might expect them to have an infinite number of zeros and decompose into factors. To some extent, this is true. According to the Picard theorem, for each entire transcendental function* f *and each* $a\in\mathbb{C}$, *except possibly one exceptional value* a, *the equation* $f(z)=a$ *has infinitely many zeros. Such an exceptional* $a\in\mathbb{C}$ *indeed exist for some entire functions. For example, for the function* $f(z)=e^z$, *the equation* $e^z=0$ *has no roots, so* $a=0$ *is an exceptional value.*

The following statement is useful when studying the expansions of holomorphic functions into series:

Theorem 2 ([24, 28, 31, 33, 45, 47, 51, 54]). *Let L be an entire function with zeros at the points λ_n and $l_n(z)=L(z)/(z-\lambda_n)$. Then for each $\delta>0$ there exists a constant $c_1(\delta)<2(2+\delta)/\delta$ such that for all $r>0$ and $n\in\mathbb{N}$, the following inequality holds:*

$$\frac{M_L(r)}{r+|\lambda_n|}\le M_{l_n}(r)\le c_1(\delta)\frac{M_L((1+\delta)r)}{r+|\lambda_n|}. \qquad (2)$$

Proof. The left-hand side of inequality (2) is evident. We will prove the right-hand side in two steps. If $\left|r-|\lambda_n|\right|\ge \varepsilon r$ where $0<\varepsilon<1$, then $M_{l_n}(r)\le M_L(r)/(\varepsilon r)$. In the case $\left|r-|\lambda_n|\right|<\varepsilon r$, putting $R=r(1+\varepsilon)/(1-\varepsilon)$, we have $\left|R-|\lambda_n|\right|=\left|(R-r)-(\lambda_n-r)\right|\ge\varepsilon r$ and consequently $M_{l_n}(r)\le M_{l_n}(R)\le M_L(R)/\delta R\le M_L(R)/\delta$. Hence,

$$rM_{l_n}(r)\le\frac{1}{\varepsilon}M_L\left(\frac{1+\varepsilon}{1-\varepsilon}r\right). \qquad (3)$$

Analogously, if $\left|r-|\lambda_n|\right|\ge\varepsilon|\lambda_n|$ then $M_{l_n}(r)\le M_L(r)/\left(\varepsilon|\lambda_n|\right)$. If $\left|r-|\lambda_n|\right|<\varepsilon|\lambda_n|$ then, by taking $r_1=r+2\varepsilon|\lambda_n|$, we get $\left|r_1-|\lambda_n|\right|\ge\varepsilon|\lambda_n|$, $|\lambda_n|\le r/(1-\varepsilon)$ and

$$M_{l_n}(r)\le M_{l_n}(r_1)\le M_L(r_1)/\varepsilon|\lambda_n|\le M_L((1+\varepsilon)r/(1-\varepsilon))/\varepsilon|\lambda_n|.$$

Thus, $|\lambda_n|M_{l_n}(r)\le M_L((1+\varepsilon)r/(1-\varepsilon))/\varepsilon$. In view of this and by using formula (3), we obtain the right-hand side of inequality (2). The theorem is proved. ►

2.10. Infinite products of real numbers. Let $(u_k):\mathbb{N}\to\mathbb{R}$ be a certain sequence of complex numbers. The symbol

$$u_1\cdot u_2\cdot\ldots\cdot u_k\cdot\ldots\cdot \qquad (1)$$

is called the product of the terms of the sequence (u_k) or an infinite product of real numbers u_k and it is denoted by [5, 14, 22, 24, 28, 45, 47, 51, 54]:

$$\prod_{k=1}^{\infty}u_k. \qquad (2)$$

This symbol is also called an infinite product. In this case,

$$p_n=\prod_{k=1}^{n}u_k=u_1\cdot u_2\cdot\ldots\cdot u_n$$

is called the n-th partial product of the product (1). If there exists the limit

$$\lim_{n\to\infty} p_n = p \neq 0;\infty\,, \tag{3}$$

then the product (1) is called convergent, and the number p is its value. This fact is written as [5, 14, 22, 24, 28, 45, 47, 51, 54]:

$$p = \prod_{k=1}^{\infty} u_k\ . \tag{4}$$

Sometimes, the sequence (p_n) of partial products is also referred to as an infinite product, as well as the operator that assigns the limit (3) to the sequence (u_k).

Example 1. *The product* $\prod_{k=1}^{\infty} 1 = 1\cdot 1\cdot \ldots$ *is convergent and* $\prod_{k=1}^{\infty} 1 = 1$, *because* $p_n = \prod_{k=1}^{n} 1 = 1\cdot 1\cdot \ldots \cdot 1 = 1$.

Example 2. *The product* $\prod_{k=1}^{\infty} (-1)^k$ *is divergent, because* $p_n = \prod_{k=1}^{n} (-1)^k = (-1)^n$ *and the limit (3) does not exist. At the same time, the product* $\prod_{k=1}^{\infty} \left|(-1)^k\right|$ *converges.*

Example 3. *The product* $\prod_{k=2}^{\infty} (1 - 1/k^2) = \prod_{m=1}^{\infty} (1 - 1/(m+1)^2)$ *is convergent and* $\prod_{k=2}^{\infty} (1 - 1/k^2) = 1/2$, *because*

$$\prod_{k=2}^{n} (1 - 1/k^2) = \prod_{k=2}^{n} \frac{k^2 - 1}{k^2} = \prod_{k=2}^{n} \frac{(k+1)(k-1)}{kk} =$$

$$= \frac{1\cdot 3}{2\cdot 2}\cdot\frac{2\cdot 4}{3\cdot 3}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{4\cdot 6}{5\cdot 5}\cdot\frac{5\cdot 7}{6\cdot 6}\cdot\frac{6\cdot 8}{7\cdot 7}\cdot \ldots \cdot\frac{(n-3)(n-1)}{(n-2)(n-2)}\cdot\frac{(n-2)n}{(n-1)(n-1)}\cdot\frac{(n-1)(n+1)}{nn} =$$

$$= \frac{(n+1)}{2n} \to \frac{1}{2}\,.$$

Theorem 1. *If the product (2) is convergent, then* $u_n \to 1$ *as* $n \to \infty$.

Proof. In fact, because $u_n = p_n / p_{n-1}$. ▶

Example 4. *The product* $\prod_{k=1}^{\infty} \frac{2k+10}{k^2}$ *is divergent, because*

$\lim\limits_{k\to\infty} \frac{2k+10}{k^2} = 0$.

Theorem 2. *The product (2) with* $u_k > 0$ *converges if and only if the series* $\sum\limits_{k=1}^{\infty} \ln u_k$ *converges.*

Proof. Indeed, it follows from the equality $p_n = \exp\left(\sum\limits_{k=1}^{n} \ln u_k\right)$, $\sum\limits_{k=1}^{n} \ln u_k = \ln p_n$. ►

Example 5. *The product* $\prod\limits_{k=1}^{\infty} (1-1/k)e^{1/k}$ *is convergent, because* $\ln\left((1-1/k)e^{1/k}\right) = \frac{1}{k} + \ln(1-1/k) = \frac{1}{2k^2} + o\left(\frac{1}{k^2}\right)$, $k \to \infty$, *and therefore the series* $\sum\limits_{k=1}^{\infty} \ln\left((1-1/k)e^{1/k}\right)$ *converges.*

Remark 1. *If one of the terms of a sequence* (u_k) *equals zero, then according to the definition, the product (1) is divergent. In the next section, we will provide a slightly more general definition of the convergence of an infinite product, so that the product can be convergent even if a finite number of its factors are zero.*

2.11. Infinite products of complex numbers. The product of the terms of a sequence $(u_k): \mathbb{N} \to \mathbb{C}$ or an infinite product of complex numbers u_k is called the symbol [5, 14, 22, 24, 28, 45, 47, 51, 54]

$$u_1 \cdot u_2 \cdot \ldots \cdot u_k \cdot \ldots \cdot, \tag{1}$$

which is also denoted as:

$$\prod_{k=1}^{\infty} u_k . \tag{2}$$

The product (1) is said to converge if for some n_0 there exists the limit

$$\lim_{n\to\infty} \prod_{k=n_0+1}^{n} u_k = p \neq 0; \infty . \tag{3}$$

In this case, the number p is called the value of infinite product (2) and this fact is written as: $p = \prod\limits_{k=1}^{\infty} u_k$.

Theorem 1. *If the product (2) converges, then* $u_n \to 1$ *as* $n \to \infty$.

Theorem 2 ([5, 14, 22, 24, 28, 45, 47, 51, 54]). *The product (2) converges if and only if the series*

$$\sum_{k=n_0+1}^{\infty} \ln u_k \ , \tag{4}$$

converges fo some $n_0 \in \mathbb{N}$ *and the value* $\ln u_k := \ln|u_k| + i\arg u_k \in \operatorname{Ln} u_k$.

Proof. Indeed, the sufficiency follows from the equality

$$p_n = \exp\left(\sum_{k=1}^{n} \ln u_k\right).$$

Let us prove the necessity. Since $u_k \to 1$ as $k \to \infty$, it is sufficient to show that the series (4) converges if, for large k, the branch of $\ln$ near the point 1 is understood in such a way that it takes the value 0 at that point. By the Cauchy convergence criterion for the sequence, we have

$$(\forall \varepsilon > 0)(\exists n^\bullet)(\forall n \geq n^\bullet)(\forall m \geq n^\bullet) : |p_n / p_m - 1| < \varepsilon ,$$

because $p_m \to p \neq 0; \infty$. Therefore, $\operatorname{Re}(p_n / p_{n^\bullet}) > 0$. If $\ln$ is the main branch of the logarithm and s_n is the n-th partial sum of the series (4), then $\ln p_n / p_{n^\bullet} = s_n - s_{n^\bullet} + 2g_n\pi i,\ g_n \in \mathbb{Z}$. Since $p_n \to p$, we have $p_n / p_{n^\bullet} \to p / p_{n^\bullet}$. Therefore, by the continuity of the principal value of the logarithm in $\mathbb{C}_+$, the sequence $\alpha_n = s_n - s_{n^\bullet} + 2g_n\pi i$ has a finite limit. The series

$$\sum_{k=n^\bullet+1}^{\infty} \ln u_k + 2\pi i\left(g_k - g_{k-1}\right)$$

is convergent, because its partial sums are equal to $s_n - s_{n^\bullet} + 2\pi i(g_n - g_{n^\bullet})$. Therefore, $g_k - g_{k-1} \to 0$. However, $g_k \in \mathbb{Z}$. Thus, $g_k = g_{k-1}$ for large k, and the series (4) converges. The theorem is proved. ►

Theorem 2 can be stated as follows: the product (2) is convergent if and only if the series (3) is convergent, where $\ln u_k := \ln|u_k| + i\arg u_k$ and $\arg u_k \in [-\pi;\pi)$ for all sufficiently large k.

An infinite product (2) is said to converge absolutely if the series (4) is absolutely convergent [5, 14, 22, 24, 28, 45, 47, 51, 54]. An absolutely convergent product is convergent. At the same time, absolute convergence of the product (2) is not equivalent to the convergence of the product $\prod_{k=1}^{\infty} |u_k|$.

Theorem 3 ([5, 14, 22, 24, 28, 45, 47, 51, 54]). *In order that the product (2) with* $u_k = 1 + a_k$ *be absolutely converges, it is necessary and sufficient that the following series converges:*

$$\sum_{k=1}^{\infty} |a_k| \quad . \tag{5}$$

Proof. If the product is absolutely convergent, then it is convergent, and therefore $a_k \to 0$, $\arg(1+a_k) \to 0$ and $\ln(1+a_k) = (1+o(1))a_k$ as $k \to \infty$. Hence, the series (5) is convergent. Conversely, if the series (5) converges, then $a_k \to 0$ and $\ln(1+a_k) = (1+o(1))a_k$ as $k \to \infty$. Thus, the product (2) is absolutely convergent. Theorem 3 is proved. ►

The product $r_p = \prod_{k=p+1}^{\infty} u_k$ is called the remainder of the product (2). If all $u_k \neq 0$, then the product (2) is convergent if and only if each of its remainders is convergent. In this case, $r_p \to 1$ as $p \to \infty$.

Example 1. *Since* $\frac{k^2}{k^2+1} = 1 - \frac{1}{k^2+1}$ *and the series* $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ *is convergent, then the product* $\prod_{k=1}^{\infty} \frac{k^2}{k^2+1}$ *is absolutely convergent.*

Example 2. *Since* $\frac{k^2}{k^2+1} = 1 - \frac{1}{k^2+1}$ *and the series* $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ *is convergent, then the product* $\prod_{k=1}^{\infty} \left| (-1)^k \frac{k^2}{k^2+1} \right|$ *is convergent. In this case, the product* $\prod_{k=1}^{\infty} (-1)^k \frac{k^2}{k^2+1}$ *diverges, because the limit* $\lim_{k\to\infty} (-1)^k \frac{k^2}{k^2+1}$ *does not exists.*

Example 3. *If* $u_1 = 2$, $u_2 = u_3 = 0$ *and* $u_k = \frac{k^2}{k^2+1}$ *for* $k \geq 3$, *then the product (2) converges and* $\prod_{k=1}^{\infty} u_k = 0$.

Example 4. *The product* $\prod_{k=1}^{\infty}\left(1+\frac{1}{k^2+i}\right)$ *is absolutely convergent, because* $\left|\frac{1}{k^2+i}\right|=\frac{1}{\sqrt{k^4+1}}$ *and the series* $\sum_{k=1}^{\infty}\frac{1}{\sqrt{k^4+1}}$ *converges.*

Example 5. *Since* $k^2\left(e^{1/k}-1\right)\to\infty$, *the product* $\prod_{k=1}^{\infty}k^2\left(e^{1/k}-1\right)$ *is divergent.*

2.12. Infinite products of functions. Let (u_k) be a sequence of functions holomorphic in the domain D. Then the infinite product

$$\prod_{k=1}^{\infty}u_k(z) \tag{1}$$

is called an infinite product of functions. For each $z\in D$ an infinite product of functions (1) is a numerical infinite product [5, 14, 22, 24, 28, 45, 47, 51, 54]. An infinite product of functions (1) is said to converge (absolutely converge) on a set $E\subset D$ if for every $z\in E$ the corresponding numerical product converges (absolutely converges). The product (1) is said to uniformly converge on a set $E\subset D$ if for some $n_0\in\mathbb{N}$ the sequence [5, 14, 22, 24, 28, 45, 47, 51, 54]

$$\pi_n(z)=\prod_{k=n_0+1}^{n}u_k(z)$$

uniformly converges on E to the function $r_{n_0}(z)$. In this case, $p(z)=r_{n_0}(z)\prod_{k=1}^{n_0}u_k(z)$ is called a value of a product (1) and this fact is written as: $p(z)=\prod_{k=1}^{\infty}u_k(z)$. Thus, the behavior of a finite number of factors u_k does not affect the nature of the convergence of the product.

Theorem 1 ([5, 14, 22, 24, 28, 45, 47, 51, 54]). *If the functions* $u_k(z)$ *are holomorphic in the domain* $D\subset\mathbb{C}$, *and for some* $n_0\in\mathbb{N}$ *and a certain choice of values of* $\ln u_k(z)$, *the series*

$$\sum_{k=n_0+1}^{\infty}\ln u_k(z) \tag{2}$$

uniformly converges on a compact set $E\subset D$, *then an infinite product (1) converges uniformly on* E.

Proof. Indeed, let $\alpha_1(z)$ is the sum of series (2), and let $\alpha(z)=e^{\alpha_1(z)}$. Then

$$\left|\alpha(z)-\prod_{k=n_0+1}^{n}u_k(z)\right|=\left|\alpha(z)\left(1-\exp\left(\sum_{k=n_0+1}^{n}\ln u_k(z)-\sum_{k=n_0+1}^{\infty}\ln u_k(z)\right)\right)\right|=$$

$$=\left|\alpha(z)\right|\left|\left(1-\exp\left(-\sum_{k=n_0+1}^{\infty}\ln u_k(z)\right)\right)\right|.$$

If the series (2) converges uniformly on E, then $u_k(z)\to 1$ as $k\to\infty$. Therefore, $\ln u_k(z)$ converges for large k to the value of the branch $\ln z=\ln|z|+i\arg z$ with $\arg z\in(-\pi/2;\pi/2)$ in the right half-plane. Consequently, α_1 and α are continuous and bounded functions on E. Finally, it remains to use the inequality $|1-e^{-w}|\le 2|w|$ for $|w|\le 1/2$. The theorem is proved. ►

Corollary 1 ([5, 14, 22, 24, 28, 45, 47, 51, 54]). *If there exists a convergent positive numerical series* $\sum_{k=1}^{\infty}b_k$ *such that at least one of the following statements are true:*

$$(\forall z\in E)(\forall k):|\ln u_k(z)|\le b_k,$$
$$(\forall z\in E)(\forall k):|a_k(z)|\le b_k,$$

then the product (1), where $u_k(z)=1+a_k(z)$*, converges absolutely and uniformly on the compact set* $E\subset D$*.*

Example 1. *Since the series* $\sum_{k=1}^{\infty}e^{-\sqrt{k}}$ *converges, the product* $\prod_{k=1}^{\infty}\left(1-ze^{-\sqrt{k}}\right)$ *converges absolutely and uniformly in every* $\overline{U(0;R)}$*, where* $0<R<+\infty$.

Example 2. *Since* $\cos\frac{z}{k}-1=(1+o(1))\frac{|z|^2}{2k^2}$ *as* $k\to\infty$ *uniformly in* z *in every closed disk* $\overline{U(0;R)}$*,* $0<R<+\infty$*, and the series* $\sum_{k=1}^{\infty}1/k^2$

converges, then the product $\prod_{k=1}^{\infty} \cos z/k$ *converges absolutely and uniformly in each such disk.*

2.13. Constructing of an entire function with prescribed zeros. Infinite product representation of an entire function. An entire function f with a sequence of zeros (λ_k) can naturally be constructed in the form $f(z) = \prod_k (1 - z/\lambda_k)$. However, this product might be divergent. Therefore, a slight modification of the construction is necessary. The functions

$$E(w; p) = \begin{cases} 1 - w, & p = 0, \\ (1-w)\exp\left(\sum_{k=1}^{p} w^k / k\right), & p \in \mathbb{N}, \end{cases}$$

are called the Weierstrass primary factors.

Theorem 1 ([24, 28, 31, 33, 45, 47, 51, 54]). *The following inequalities are valid:*

$$\left|\ln E(w; p)\right| \le 2|w|^{p+1}, \ |w| \le 1/2, \tag{1}$$

$$\left|\ln\left|\sum_{k=1}^{p} w^k / k\right|\right| \le \left(2|w|\right)^p, \ |w| \ge 1/2, \tag{2}$$

where $\ln z$ *is the branch of the logarithm in the right half-plane that takes the value* 0 *at the point* $z = 1$.

Proof. Indeed, (1) follows from the inequalities:

$$\left|\ln E(w; p)\right| = \left|-\sum_{k=p+1}^{\infty} \frac{w^k}{k}\right| \le \sum_{k=p+1}^{\infty} |w|^k \le |w|^{p+1} \sum_{k=0}^{\infty} 2^{-k} = 2|w|^{p+1}, \ |w| \le 1/2 .$$

In addition, using $\exp\left(-|z|\right) \le |\exp(z)| \le \exp\left(|z|\right)$, we obtain

$$\left|\ln\left|\exp\left(\sum_{k=1}^{p} \frac{w^k}{k}\right)\right|\right| \le \sum_{k=1}^{p} |w|^k / k \le |w|^p \sum_{k=1}^{p} \frac{1}{2^{p-k}} \le \left(2|w|\right)^p, \ |w| \ge 1/2 .$$

The theorem is proved. ►

Theorem 2 ([24, 28, 31, 33, 45, 47, 51, 54]). *Let* (λ_k) *be a sequence of nonzero complex numbers such that* $0 < |\lambda_k| \nearrow +\infty$ *as* $k \to \infty$, *and let* (p_k) *be a sequence of nonnegative integers such that the series*

$$\sum_{k=1}^{\infty}\left(\frac{r}{|\lambda_k|}\right)^{p_k+1} \tag{3}$$

converges for every $r>0$*. Then the product*

$$L(z)=\prod_{k=1}^{\infty}E(z/\lambda_k;p_k)=\prod_{k=1}^{\infty}\left(1-\frac{z}{\lambda_k}\right)e^{\sum_{j=1}^{p_k}\frac{1}{j}\left(\frac{z}{\lambda_k}\right)^j} \tag{4}$$

converges uniformly on every compact set in $\mathbb{C}$*, the function* L *is an entire function and the sequence* (λ_k) *is a sequence of zeros of* L*.*

Proof. In according of Theorem 1, we have

$$|\ln E(z/\lambda_k;p_k)|\le 2(r/|\lambda_k|)^{p_k+1},\ |z|\le r,\ |\lambda_k|\ge 2r\ .$$

Therefore, the product (4) converges uniformly on every disk $\overline{U(0;r)}$, $0<r<+\infty$. Hence, the function L is an entire function. Theorem 2 is proved. ►

Corollary 1 ([24, 28, 31, 33, 45, 47, 51, 54]). *For every sequence* (λ_k) *that has a unique accumulation point at infinity, there exists an entire function for which* (λ_k) *is a sequence of zeros. In particular, such a function is given by the product*

$$f(z)=z^m\prod_{\lambda_k\ne 0}\left(1-\frac{z}{\lambda_k}\right)e^{\sum_{j=1}^{p_k}\frac{1}{j}\left(\frac{z}{\lambda_k}\right)^j}\ ,$$

where m *is the number of terms of the sequence* (λ_k) *that are equal to zero, and* (p_k) *is a sequence of nonnegative integers such that the series (3) converges for all* $r>0$*.*

Remark 1. *If there exists an entire function* f *for which* (λ_k) *is the sequence of its zeros, then there are infinitely many such functions. For example, functions of the form* $f_1(z)=e^z f(z)$*,* $f_2(z)=e^{\sin z}f(z)$ *and others.*

Example 1. *Let* $\lambda_k=k^2$*. Then the series (3) converges for all* $p_k=0$*, and* $L(z)=\prod_{k=1}^{\infty}(1-z/k^2)$ *is an entire function for which the sequence* (λ_k) *is a sequence of its zeros.*

Example 2. *Let* $\lambda_k=k$*. Then the series (3) converges for all* $p_k=1$*,* *and* $L(z)=\prod_{k=1}^{\infty}(1-z/k)e^{z/k}$ *is an entire function for which the sequence* (λ_k) *is a sequence of its zeros.*

Example 3. *If* $\lambda_k=\sqrt[3]{k}$, *then the series (3) converges for all* $p_k=3$ *and* $L(z)=\prod_{k=1}^{\infty}(1-z/\sqrt[3]{k})\exp\left(\frac{z}{\sqrt[3]{k}}+\frac{1}{2}\left(\frac{z}{\sqrt[3]{k}}\right)^2+\frac{1}{3}\left(\frac{z}{\sqrt[3]{k}}\right)^3\right)$ *is an entire function with a sequence of zeros* (λ_k).

Example 4. *Let* (λ_k) *be an arbitrary sequence of nonzero complex numbers such that* $0<|\lambda_k|\nearrow+\infty$. *Then the series (3) converges for every* $r>0$ *and* $p_k=k-1$, *where* $L(z)=\prod_{k=1}^{\infty}E(z/\lambda_k;k-1)$ *is an entire function for which the sequence* (λ_k) *is a sequence of its zeros.*

Theorem 3 (Weierstrass). *An entire function* $f\neq 0$ *may be represented in the form*

$$f(z)=z^m e^{g(z)}\prod_{\lambda_k\neq 0}E(z/\lambda_k;p_k)=z^m e^{g(z)}\prod_{\lambda_k\neq 0}\left(1-\frac{z}{\lambda_k}\right)\exp\left(\sum_{j=1}^{p_k}\frac{z^j}{j\lambda_k^j}\right), \quad (5)$$

where g *is an entire function,* λ_k *are all nonzero roots of* f, m *is the multiplicity of the root at the origin,* p_k *are nonnegative integers such that the series (3) converges for all* $r\geq 0$ *(if* f *has a finite number of zeros, then all* $p_k=0$*).*

Proof. Indeed, according to the uniqueness theorem, if a function $f\not\equiv 0$ has infinitely many zeros, then the set of zeros of the function f must have a unique accumulation point at infinity. Let us denote the product on the right-hand side of equality (5) by $L(z)$. Then, the function $h(z)=f(z)/z^m L(z)$ is entire and has no zeros in $\mathbb{C}$. Let

$$g(z)=\int_0^z\frac{h'(t)}{h(t)}dt\,.$$

Then $h(z)=\exp(g(z))$, which proves Theorem 3. ►

2.14. Self-control questions.

1. Formulate the definition of an entire function.
2. Formulate the definition of the maximum of the modulus of an entire function.
3. Formulate and prove Liouville's theorem.

4. Formulate the definition of an entire transcendental function.
5. Formulate and prove the Hadamard Three Circle Theorem.
6. Formulate the definition of the maximum term of an entire function.
7. Formulate the definition of the central index of an entire function.
8. Describe the construction of Newton's polygon.
9. Formulate and prove the theorem on finding the maximum term and central index.
10. Formulate and prove the theorem about the relationship between the maximum term and the central index.
11. Formulate and prove the theorem about the relationship between the maximum term and the maximum of the modulus.
12. Formulate and prove the theorem about inequalities between the maximum term and the maximum of the modulus.
13. Write and prove the formula for finding the maximum of the modulus of an entire function with positive Taylor coefficients.
14. Formulate the definition of an Φ-type of an entire function.
15. Prove the theorem on finding Φ-type of an entire function.
16. Formulate the theorem on the approximation of a convex function by the maximum term of an entire function.
17. Write the Poisson and Schwarz formulas.
18. Write Jensen's equality.
19. Formulate the definition of an infinite product, its convergence and absolute convergence.
20. Formulate and prove the theorem on the necessary condition for the convergence of an infinite product.
21. Formulate and prove a criterion for the convergence of an infinite product.
22. Formulate and prove a criterion for the absolute convergence of an infinite product.
23. Formulate the definition of an infinite product of functions, its pointwise, uniform and absolute convergence.
24. Formulate and prove the theorem on sufficient conditions for the absolute and uniform convergence of an infinite product of functions.
25. Formulate the definition of a Weierstrass primary factor.
26. Formulate and prove the Weierstrass theorem on the expansion of an entire function into infinite product.

2.15. Exercises and problems.

1. Determine if a given function f is entire:

1. $f(z)=2z+1$.
2. $f(z)=\cos z$.
3. $f(z)=z+1/z$.
4. $f(z)=\dfrac{z-1}{z+1}$.

5. $f(z)=z^n,\ n\in\mathbb{N}$.

6. $f(z)=1/z$.

7. $f(z)=\sin z$.

8. $f(z)=e^z$.

9. $f(z)=\overline{z}$.

10. $f(z)=\operatorname{tg} z$.

11. $f(z)=\operatorname{Re} z^2$.

12. $f(z)=(\operatorname{Re} z)^2$.

13. $f(z)=\overline{z^2}$.

14. $f(z)=|z|$.

15. $f(z)=\sum_{k=1}^{\infty}\frac{2^k}{(1+1/k)^{k^2}}z^{2k}$.

16. $f(z)=\sum_{k=0}^{\infty}\frac{2^k}{(1+k)^k}z^{k-1}$.

2. Find the maximum of the modulus of an entire function f:

1. $f(z)=1+z^2$.

2. $f(z)=e^{-z^2}$.

3. $f(z)=e^{-2z^3}$.

4. $f(z)=1-3z^3$.

5. $f(z)=e^{-3z^2}+z^2$.

6. $f(z)=e^{z^2}+1+z$.

7. $f(z)=e^{-2z^3}-z$.

8. $f(z)=e^{2z^4}-3z^3$.

9. $f(z)=e^{-z^3}-2z$.

10. $f(z)=e^{(1+2i)z^3}$.

11. $f(z)=\sin z$.

12. $f(z)=\cos z$.

13. $f(z)=e^{\sin z}$.

14. $f(z)=e^{(1-3i)z^2}$.

3. Prove that (here $z=x+iy$):

1. $\operatorname{Re}\sin z=\sin x\operatorname{ch} y$.
2. $\operatorname{Im}\sin z=\cos x\sin y$.
3. $\operatorname{Re}\cos z=\cos x\operatorname{ch} y$.
4. $\operatorname{Im}\sin z=-\sin x\operatorname{sh} y$.
5. $|\sin z|=\sqrt{\operatorname{ch}^2 y-\cos^2 x}$.
6. $|\operatorname{ch} z|=\sqrt{\operatorname{ch}^2 x-\sin^2 y}$.
7. $|\cos z|=\sqrt{\operatorname{ch}^2 y-\sin^2 x}$.
8. $|\operatorname{sh} z|=\sqrt{\operatorname{ch}^2 x-\cos^2 y}$.
9. $\operatorname{sh}(|\operatorname{Im} z|)\le|\sin z|\le\operatorname{ch}(|\operatorname{Im} z|)$.
10. $e^{-|z|}\le e^z\le e^{|z|},\ z\in\mathbb{C}$.
11. $\operatorname{sh}(|\operatorname{Im} z|)\le|\cos z|\le\operatorname{ch}(|\operatorname{Im} z|)$.
12. $|\sin r|\le|\sin z|\le\operatorname{sh} r,\ |z|=r$.

13. $|\cos r| \le |\cos z| \le \operatorname{ch} r$, $|z| = r$.

14. $\dfrac{2e^{-2\operatorname{Im} z}}{1+e^{-2\operatorname{Im} z}} < |i - \operatorname{tg} z| < \dfrac{2e^{-2\operatorname{Im} z}}{1-e^{-2\operatorname{Im} z}}$, $\operatorname{Im} z > 0$.

15. $\dfrac{2e^{-2\operatorname{Im} z}}{1+e^{-2\operatorname{Im} z}} < |i + \operatorname{tg} z| < \dfrac{2e^{-2\operatorname{Im} z}}{1-e^{-2\operatorname{Im} z}}$, $\operatorname{Im} z < 0$.

16. $\dfrac{2e^{-2\operatorname{Im} z}}{1+e^{-2\operatorname{Im} z}} < |i - \operatorname{ctg} z| < \dfrac{2e^{-2\operatorname{Im} z}}{1-e^{-2\operatorname{Im} z}}$, $\operatorname{Im} z < 0$.

17. $\dfrac{2e^{-2\operatorname{Im} z}}{1+e^{-2\operatorname{Im} z}} < |i + \operatorname{ctg} z| < \dfrac{2e^{-2\operatorname{Im} z}}{1-e^{-2\operatorname{Im} z}}$, $\operatorname{Im} z > 0$.

4. Find the Φ-type of an entire function f:

1. $f(z) = 1 + z$, $\Phi(z) = 1 + z$.
2. $f(z) = e^z$, $\Phi(z) = e^{2z}$.
3. $f(z) = e^z$, $\Phi(z) = 1 + 2z$.
4. $f(z) = e^z$, $\Phi(z) = e^{z^2}$.
5. $f(z) = e^z$, $\Phi(z) = e^{e^z}$.
6. $f(z) = z$, $\Phi(z) = e^z$.
7. $f(z) = \sum_{k=0}^{\infty} \dfrac{z^k}{(2k+1)^k}$, $\Phi(z) = \sum_{k=0}^{\infty} \dfrac{z^k}{(k+1)^k}$.
8. $f(z) = \sum_{k=0}^{\infty} \dfrac{z^k}{(3k+1)^k}$, $\Phi(z) = \sum_{n=0}^{\infty} \dfrac{2^k}{k!} z^k$.
9. $f(z) = \sum_{k=0}^{\infty} \dfrac{e^k}{k!} z^k$, $\Phi(z) = \sum_{k=0}^{\infty} \dfrac{z^k}{(2k)!}$.
10. $f(z) = \sum_{k=0}^{\infty} \dfrac{z^k}{k!}$, $\Phi(z) = \sum_{k=0}^{\infty} \dfrac{z^k}{k^k}$.

5. Investigate on convergence the infinite products:

1. $\prod_{n=1}^{\infty} e^{\frac{1}{n}}$.
2. $\prod_{n=1}^{\infty} e^{\frac{1}{n^2}}$.
3. $\prod_{n=1}^{\infty} \dfrac{n^2 + i}{n^2 + 2i}$.
4. $\prod_{n=1}^{\infty} \dfrac{e^{\frac{1}{n}}}{1 + \frac{1}{n}}$.

5. $\prod_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{1+\frac{1}{n^2}}$.

6. $\prod_{n=1}^{\infty} \frac{in^3+3i}{in^3+2}$.

7. $\prod_{n=1}^{\infty} \left(1-\frac{1}{n}\right)$.

8. $\prod_{n=1}^{\infty} \left(1-\frac{1}{n}\right) e^{\frac{1}{n}}$.

9. $\prod_{n=1}^{\infty} \left(1-\frac{1}{n}\right) e^{\frac{1}{n}+\frac{1}{2n^2}}$.

10. $\prod_{n=1}^{\infty} \left(1-\frac{2i}{1+n}\right)$.

6. Investigate on absolute and uniform convergence the infinite products of functions:

1. $\prod_{k=1}^{\infty} \left(1+\frac{z}{k\ln^2(k+1)}\right)$.

2. $\prod_{n=1}^{\infty} \left(1+\frac{z}{2^n}\right)$.

3. $\prod_{n=1}^{\infty} \left(1+z\ln\left(1+\frac{1}{n^{3/2}}\right)\right)$.

4. $\prod_{n=1}^{\infty} \left(1-z\left(1-\cos\frac{1}{n}\right)\right)$.

5. $\prod_{k=1}^{\infty} \left(1+\frac{z}{(k+1)\ln(k+1)}\right) e^{-\frac{z}{(k+1)\ln(k+1)}}$.

6. $\prod_{n=1}^{\infty} \left(1+\frac{z}{e^{\sqrt{n}}}\right)$.

7. $\prod_{n=1}^{\infty} \left(1+z\left(\sin\frac{1}{n}-\frac{1}{n}\right)\right)$.

8. $\prod_{k=1}^{\infty} \left(1-\frac{z}{\sqrt{k}}\right) e^{\frac{z}{\sqrt{k}}+\frac{z^2}{2k}}$.

9. $\prod_{n=1}^{\infty} \left(1-z\left(1-\cos\frac{1}{n}\right)\right)$.

10. $\prod_{n=1}^{\infty} \left(1+z\left(\sin\frac{1}{n}-\frac{1}{n}\right)\right)$.

11. $\prod_{k=1}^{\infty} \left(1+\frac{z}{k\ln^2(k+1)}\right) e^{-\frac{z}{(k+1)\ln^2(k+1)}}$.

12. $\prod_{n=1}^{\infty} \left(1-2^n z\right)$.

7. Prove the statements:

1. $\prod_{k=0}^{\infty} \left(1+\frac{1}{4^k}\right)=2$.

2. $\prod_{k=0}^{\infty} \left(1+x^{2k}\right)=\frac{1}{1-x}$.

3. $\prod_{k=1}^{\infty} \cos\frac{x}{2^k}=\frac{\sin x}{x}$.

4. $\prod_{k=1}^{\infty} \cos\frac{\pi}{2^{k+1}}=\frac{2}{\pi}$.

5. $\prod_{k=2}^{\infty} \left(1-\frac{2}{k(k+1)}\right)=\frac{1}{3}$.

6. $\prod_{k=1}^{\infty} \frac{9k^2}{9k^2-1}=\frac{2\pi}{3\sqrt{3}}$.

7. $\dfrac{\prod_{k=1}^{\infty}(1-q^{2k})}{\prod_{k=1}^{\infty}(1-q^{2k-1})}=\sum_{k=0}^{\infty}q^{k(k+1)/2}$.

8. $\int_0^1 x^{-x}dx=\sum_{n=1}^{\infty}n^{-n}$.

9. $f(x)=(1-qx)f(qx)$ if $f(x)=\prod_{k=1}^{\infty}(1-q^k x)$, $x\in\mathbb{R}$, $|q|<1$.

10. $\prod_{k=1}^{\infty}(1-q^k x)=\sum_{k=0}^{\infty}\dfrac{q^{k(k+1)/2}}{\prod_{j=1}^{k}(q^j-1)}x^k$, $x\in\mathbb{R}$, $|q|<1$.

11. $\dfrac{1}{\prod_{k=1}^{\infty}(1-q^k x)}=\sum_{k=0}^{\infty}\dfrac{q^k}{\prod_{j=1}^{k}(1-q^j)}x^k$, $x\in\mathbb{R}$, $|q|<1$.

12. $\prod_{k=1}^{\infty}(1-q^k)=\sum_{k=-\infty}^{\infty}(-1)^k q^{(3k^2+k)/2}$, $|q|<1$.

8. Find the zeros of a function f and their orders:

1. $f(z)=(z^2-\pi^2)\sin z$.
2. $f(z)=z^2\sin z$.
3. $f(z)=(z-\pi)^2\sin z$.
4. $f(z)=\sin z+\sin^2 z$.
5. $f(z)=z(e^z-1)$.
6. $f(z)=e^{z^2}-1$.
7. $f(z)=\dfrac{\sin^3 z}{z^2}$.
8. $f(z)=\sin 3z-3\sin z$.
9. $f(z)=e^{z-1}-1$.
10. $f(z)=e^{2z}-3e^z+2$.
11. $f(z)=z^3-z^2-8z+12$.
12. $f(z)=z\,\mathrm{sh}\,z$.

9. Find the counting functions $n(r)$ and $N(r)$ of a sequence (λ_n) if:

1. $\lambda_n=n-\dfrac{1}{4}$, $n\in\mathbb{N}$.
2. $\lambda_n=\dfrac{2}{7}n$, $n\in\mathbb{N}$.
3. $\lambda_n=3\sqrt[5]{n}$, $n\in\mathbb{N}$.
4. $\lambda_n=i\sqrt[3]{n}$, $n\in\mathbb{N}$.
5. $\lambda_n=2^{n-1}$, $n\in\mathbb{N}$.
6. $\lambda_n=e^n$, $n\in\mathbb{N}$.

10. Let (λ_k) be a sequence of points on the circle $U(0;1)$ satisfying $\sum_k(1-|\lambda_k|)<+\infty$. Prove that the Blaschke product

$$B(z)=\prod_k \frac{\lambda_k - z}{1-\bar{\lambda}_k z}\frac{\bar{\lambda}_k}{\lambda_k}$$

uniformly and absolutely converges on every compact set from $U(0;1)$ and $|B(z)|\le 1$.

2.16. Individual tasks.

1. Determine if a given function f is entire:

1. $f(z)=e^{e^z}$.
2. $f(z)=\cos\sqrt{z}$.
3. $f(z)=e^{z^2}$.
4. $f(z)=\dfrac{\ln(1+z)}{z}$.
5. $f(z)=\dfrac{\sin z}{z^2-\pi z}$.
6. $f(z)=\dfrac{\sin\sqrt{z}}{\sqrt{z}}$.
7. $f(z)=\sum\limits_{k=0}^{\infty}\dfrac{2^k}{(1+k)!}(z-1)^k$.
8. $f(z)=\sum\limits_{k=0}^{\infty}\dfrac{2^k}{k!}z^k$.
9. $f(z)=\sum\limits_{n=0}^{\infty}n\dfrac{i+n}{2i+n}z^n$.
10. $f(z)=\sum\limits_{n=1}^{\infty}2^n\dfrac{i+n}{1+n}z^{2n}$.
11. $f(z)=\sum\limits_{k=1}^{\infty}3^k i^k (z+i)^{3k}$.
12. $f(z)=\sum\limits_{k=0}^{\infty}\ln(1+e^{-k^2})(z+i)^{3k}$.
13. $f(z)=\sum\limits_{k=1}^{\infty}\dfrac{e^{2ik}}{k^3}z^k$.
14. $f(z)=\sum\limits_{k=1}^{\infty}\dfrac{2^{2k-2}}{(2k-1)!}z^{2k-1}$.
15. $f(z)=\sum\limits_{k=0}^{\infty}\dfrac{z^{2k}}{k!}$.
16. $f(z)=\sqrt{z}\sin z$.
17. $f(z)=\dfrac{\ln(1+z)}{z}$.
18. $f(z)=\sum\limits_{k=1}^{\infty}\dfrac{z^k}{k^{\ln k}}$.
19. $f(z)=\sum\limits_{k=1}^{\infty}\left(\dfrac{z}{k}\right)^k$.
20. $f(z)=\sum\limits_{n=0}^{\infty}e^{-n^2}z^n$.
21. $f(z)=\sum\limits_{n=0}^{\infty}\dfrac{\operatorname{ch}\sqrt{n}}{n!}z^n$.
22. $f(z)=\sum\limits_{n=2}^{\infty}\left(\dfrac{1}{n\ln n}\right)^n z^n$.
23. $f(z)=\sum\limits_{n=1}^{\infty}\dfrac{z^n}{n^{n^2}}$.
24. $f(z)=\sum\limits_{n=1}^{\infty}\left(\dfrac{\ln n}{n}\right)^n z^n$.
25. $f(z)=\sum\limits_{n=0}^{\infty}\dfrac{z^n}{\Gamma(2+n)}$.
26. $f(z)=\sum\limits_{k=0}^{\infty}\left(\dfrac{k}{e}\right)^k z^k$.

27. $f(z)=\sqrt{z}\sin\sqrt{z}$.

28. $f(z)=\dfrac{1-\cos z}{z^2}$.

29. $f(z)=\int\limits_0^z \dfrac{\sin t}{t}dt$.

30. $f(z)=\sum\limits_{k=0}^{\infty}\dfrac{k!}{2^{k^2}}z^k$.

2. Investigate on convergence the infinite products:

1. $\prod\limits_{n=1}^{\infty}\left(1-\dfrac{1}{n}\right)e^{-\frac{1}{n}}$.

2. $\prod\limits_{n=1}^{\infty}\left(1+\dfrac{1}{n}\right)e^{-\frac{1}{n}}$.

3. $\prod\limits_{k=1}^{\infty}\left(1+\dfrac{1}{\sqrt{k}}\right)e^{-\frac{1}{\sqrt{k}}}$.

4. $\prod\limits_{n=1}^{\infty}\left(1-\dfrac{1}{\sqrt{n}}\right)e^{-\frac{1}{\sqrt{n}}}$.

5. $\prod\limits_{n=1}^{\infty}\left(1+\dfrac{1}{\sqrt{n}}\right)e^{-\frac{1}{\sqrt{n}}+\frac{1}{2n}}$.

6. $\prod\limits_{n=1}^{\infty}\left(1-\dfrac{1}{\sqrt{k}}\right)e^{\frac{1}{\sqrt{k}}+\frac{1}{2k}}$.

7. $\prod\limits_{n=1}^{\infty}\left(1+\dfrac{1}{\sqrt{n}}\right)e^{-\frac{1}{\sqrt{n}}+\frac{1}{2n}-\frac{1}{3n\sqrt{n}}}$.

8. $\prod\limits_{n=1}^{\infty}\sqrt[n]{1+\dfrac{1}{n}}$.

9. $\prod\limits_{n=1}^{\infty}\left(1-\dfrac{1}{n^2}\right)e^{\frac{1}{n}}$.

10. $\prod\limits_{n=1}^{\infty}\left(1+\dfrac{1}{n}\right)e^{-\frac{1}{n^2}}$.

11. $\prod\limits_{n=1}^{\infty}\dfrac{n^2+i}{n^2+2i}$.

12. $\prod\limits_{n=2}^{\infty}\dfrac{e^{-\frac{1}{n}}}{1-\frac{1}{n^2}}$.

13. $\prod\limits_{n=1}^{\infty}\left(1-\dfrac{1}{n}\right)e^{\frac{1}{n}+\frac{1}{2n^2}}$.

14. $\prod\limits_{k=2}^{\infty}\dfrac{e^{\frac{1}{k}}}{1+\frac{1}{k^2}}$.

15. $\prod\limits_{k=1}^{\infty}\cos^2\dfrac{1}{k}$.

16. $\prod\limits_{n=1}^{\infty}\dfrac{in^2-\sqrt{5}i}{-2in^2+3}$.

17. $\prod\limits_{n=1}^{\infty}\dfrac{k^4+2}{k^4+1}$.

18. $\prod\limits_{n=1}^{\infty}\dfrac{2^k+2}{2^k+1}$.

19. $\prod\limits_{n=1}^{\infty}\sqrt{\dfrac{n+1}{n+3}}$.

20. $\prod\limits_{n=1}^{\infty}\dfrac{n^2+2}{n+\sin\frac{1}{n}}$.

21. $\prod\limits_{n=1}^{\infty}\dfrac{i+2^n}{2i+3^n}$.

22. $\prod\limits_{n=1}^{\infty}\left(1+\dfrac{i}{1+n^2}\right)$.

23. $\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1}$.

24. $\prod_{n=3}^{\infty} \frac{n^2-4}{n^2-1}$.

25. $\prod_{n=1}^{\infty} \cos\frac{1}{2^n}$.

26. $\prod_{n=2}^{\infty} \int_n^{n+1} \frac{x^2}{1+x^2}dx$.

27. $\prod_{n=1}^{\infty} \left(1-\frac{1}{(n+1)^2}\right)$.

28. $\prod_{n=1}^{\infty} \left(1+\frac{(-1)^{n+1}}{n}\right)$.

29. $\prod_{n=1}^{\infty} n^{(-1)^n}$.

30. $\prod_{n=1}^{\infty} \frac{n-4i}{\sqrt{n}+2i}$.

3. Find the zeros of a function f and their orders:

1. $f(z)=\frac{z^2+9}{z^4}$.

2. $f(z)=z^2\sin z$.

3. $f(z)=z\cos^2 z$.

4. $f(z)=(z^2+2z+1)(e^z-1)$.

5. $f(z)=(z^2+1)^3\operatorname{tg} z$.

6. $f(z)=\frac{\sin^3 z}{z^2}$.

7. $f(z)=\frac{(1+z^2)^2}{1-z^2}$.

8. $f(z)=\frac{1}{z^2}e^{\frac{1}{z+1}}$.

9. $f(z)=\cos z\operatorname{ch} z$.

10. $f(z)=\sin 5z-5\sin z$.

11. $f(z)=z\operatorname{tg}^2 z$.

12. $f(z)=e^{2z}-3e^z+2$.

13. $f(z)=z\operatorname{sh} z$.

14. $f(z)=z^3-z^2-8z+12$.

15. $f(z)=e^{z-1}-1$.

16. $f(z)=\sin z+\sin^2 z$.

17. $f(z)=\sin(z-1)\cos^3\frac{\pi z}{2}$.

18. $f(z)=\frac{\sin^2(z-1)}{\cos(\pi z/2)}$.

19. $f(z)=\frac{1-\cos z^2}{1+\cos z}$.

20. $f(z)=\frac{1}{z}\sin\frac{1}{z}$.

21. $f(z)=e^{\frac{1}{z}}-1$.

22. $f(z)=z(e^z-1)$.

23. $f(z)=\frac{(z^2-z-2)^3}{1+\cos(\pi z)}$.

24. $f(z)=\frac{(z^2-9)^2(z-1)^3}{z^2-4z+3}$.

25. $f(z)=(z-1)^7(1-\cos z)$.

26. $f(z)=z^3\sin z$.

27. $f(z)=\cos z^3$.

28. $f(z)=\cos^3 z$.

29. $f(z)=\cos(z-\pi)$.

30. $f(z)=\cos(z-i)$.

4. Construct an entire function f for which the sequence $(\lambda_k)_{k\in\mathbb{N}}$ is a sequence of its zeros:

1. $\lambda_k = k^{\sqrt{2}}$.
2. $\lambda_k = k$.
3. $\lambda_k = \sqrt{k}$.
4. $\lambda_k = \sqrt[4]{k}$.
5. $\lambda_k = \ln k$, $k \in \mathbb{N} \setminus \{1\}$.
6. $\lambda_k = e^{\sqrt{k}}$.
7. $\lambda_k = 1 + i\sqrt{k}$.
8. $\lambda_k = k^2 + ik^4$.
9. $\lambda_k = k^{2/3} + ik^{1/2}$.
10. $\lambda_k = k^{1/3} + ik^{1/4}$.
11. $\lambda_k = k^3$.
12. $\lambda_k = \ln^2 k$, $k \in \mathbb{N} \setminus \{1\}$.
13. $\lambda_k = k \ln^2 k$, $k \in \mathbb{N} \setminus \{1\}$.
14. $\lambda_k = e^{-k}$.
15. $\lambda_k = \sqrt{k^5}$.
16. $\lambda_k = i\sqrt[3]{k^4}$.
17. $\lambda_k = k^3 + i\sqrt{k}$.
18. $\lambda_k = (1+2i)\sqrt{k}$.
19. $\lambda_k = k \ln k$.
20. $\lambda_k = \sqrt{k} + i$.
21. $\lambda_k = k^2 \ln k$.
22. $\lambda_k = 2\sqrt{k} + ki$.
23. $\lambda_k = 2^k$.
24. $\lambda_k = k^2 i$.
25. $\lambda_k = \sqrt{k+4} - ki$.
26. $\lambda_k = k - 7i$.
27. $\lambda_k = 2\pi ki$.
28. $\lambda_k = \pi ki$.
29. $(1;1;1;2^2;2^2;2^2;3^2;3^2;3^2;\ldots;n^2;n^2;n^2;\ldots)$.
30. $\lambda_k = k^2 + k + i$.

Chapter 3. Growth characteristics of entire functions

3.1. Order and type of entire functions. The order of an entire function

$$f(z)=\sum_{k=0}^{\infty} f_k z^k \qquad (1)$$

is called a number $\rho=\rho_f=\rho[f]$ determined by the formula:

$$\rho=\overline{\lim_{r\to+\infty}}\frac{\ln^+\ln^+ M_f(r)}{\ln r}, \quad \ln^+ a:=\begin{cases}\ln a, & a\ge 1,\\ 0, & 0\le a<1.\end{cases}$$

In other words, the order of an entire function f is the greatest lower bound of those values of $\rho_1\in[0;+\infty]$ for which $(\exists c_1)(\forall z\in\mathbb{C}):\left|f(z)\right|\le c_1\exp\left(|z|^{\rho_1}\right)$, that is $(\exists r_0\in[0;+\infty))(\forall z,|z|\ge r_0):\left|f(z)\right|\le\exp\left(|z|^{\rho_1}\right)$. The order of an entire function f is equal to zero if and only if

$$(\forall\rho_1\in(0;+\infty))(\exists r_0\in[0;+\infty))(\forall z,|z|\ge r_0):\left|f(z)\right|\le\exp\left(|z|^{\rho_1}\right).$$

The order of the function f is equal to $+\infty$ if and only if

$$(\forall\rho_1\in(0;+\infty))(\exists z\in\mathbb{C}):\left|f(z)\right|\ge\exp\left(|z|^{\rho_1}\right).$$

The order of the function f is equal to the number $\rho\in(0;+\infty)$ if and only if the following conditions are fulfilled:

1) $(\forall\rho_1>\rho)(\exists r_0\in[0;+\infty))(\forall z,|z|\ge r_0):\left|f(z)\right|\le\exp\left(|z|^{\rho_1}\right)$;

2) there exists a sequence (z_k), $z_k\to\infty$, such that

$$(\forall\rho_2<\rho)(\exists k_0\in\mathbb{N})(\forall k\ge k_0):\left|f(z_k)\right|\ge\exp\left(|z_k|^{\rho_2}\right).$$

If $\rho\in(0;+\infty)$ be the order of an entire function f, then the number $\sigma=\sigma[f]=\sigma[f;\rho]$, defined by the formula

$$\sigma=\overline{\lim_{r\to\infty}}\frac{\ln^+ M_f(r)}{r^{\rho}}, \qquad (2)$$

is called the type of the function f.

The type of an function f of order $\rho\in(0;+\infty)$ is equal to zero if and only if

$$\left(\forall\sigma_1\in(0;+\infty)\right)\left(\exists r_0\in[0;+\infty)\right)\left(\forall z,|z|\ge r_0\right):\left|f(z)\right|\le\sigma_1\exp\left(|z|^{\rho}\right).$$

The order of the function f of order $\rho\in(0;+\infty)$ is equal to $+\infty$ if and only if there exists a sequence (z_k), $z_k\to\infty$, such that $\left(\forall\sigma_2\in(0;+\infty)\right)\left(\exists k_0\in\mathbb{N}\right)\left(\forall k\ge k_0\right):\left|f(z_k)\right|\ge\sigma_2\exp\left(|z_k|^{\rho}\right)$. The type of the function f is equal to the number $\sigma\in(0;+\infty)$ if and only if the following conditions are valid:

1) $\left(\forall\sigma_1>\sigma\right)\left(\exists r_0\in[0;+\infty)\right)\left(\forall z,|z|\ge r_0\right):\left|f(z)\right|\le\sigma_1\exp\left(|z|^{\rho}\right)$;

2) there exists a sequence (z_k), $z_k\to\infty$, such that $\left(\forall\sigma_2<\sigma\right)\left(\exists k_0\in\mathbb{N}\right)\left(\forall k\ge k_0\right):\left|f(z_k)\right|\ge\sigma_2\exp\left(|z_k|^{\rho}\right)$.

If $\sigma=0$, $0<\sigma<+\infty$ and $\sigma=+\infty$, then f is an entire function of minimal, normal or maximal type, respectively.

Remark 1. *Sometimes, it is useful to consider the type* $\sigma=\overline{\lim_{r\to\infty}}\, r^{-\rho_1}\ln^{+}M_f(r)$ *of a function* f *with respect to an arbitrary number* $\rho_1\in(0;+\infty)$*, which does not necessarily equal its order. In this case, the type of the function* f *is equal to* σ *with respect to the formal order* ρ_1*. Typically, the formal order considered is not less than the usual order of the function.*

Remark 2. *If the function* f *is not constant, then*

$$\rho=\overline{\lim_{r\to+\infty}}\frac{\ln\ln M_f(r)}{\ln r}\quad\text{and}\quad\sigma=\overline{\lim_{r\to+\infty}}\frac{\ln M_f(r)}{r^{\rho}}.$$

Example 1. *If* $f(z)=e^{z}$*, then*

$$\left|f(re^{i\theta})\right|=\left|e^{re^{i\theta}}\right|=\left|e^{r\cos\theta+ir\sin\theta}\right|=e^{r\cos\theta}\left|\cos\left(r\sin\theta\right)+i\sin(r\sin\theta)\right|=e^{r\cos\theta},$$

$M_f(r)=e^{r}$ *and* $\rho=\lim_{r\to+\infty}\dfrac{\ln\ln M_f(r)}{\ln r}=1.$

Example 2. *If* $f(z)=e^{\tau z^{n}}$*, where* $\tau=se^{i\psi}\in\mathbb{C}$ *and* $n\in\mathbb{N}$*, then*

$$\left|f(re^{i\theta})\right| = \left|e^{se^{i\psi}r^n e^{in\theta}}\right| = \left|e^{sr^n e^{i(n\theta+\psi)}}\right| = \left|e^{sr^n(\cos(n\theta+\psi)+i\sin(n\theta+\psi))}\right| =$$

$$= \left|e^{sr^n\cos(n\theta+\psi)}\left(\cos\left(sr^n\sin(n\theta+\psi)\right)+i\sin\left(sr^n\sin(n\theta+\psi)\right)\right)\right| =$$

$$= e^{sr^n\cos(n\theta+\psi)}\left|\left(\cos\left(sr^n\sin(n\theta+\psi)\right)+i\sin\left(sr^n\sin(n\theta+\psi)\right)\right)\right| = e^{sr^n\cos(n\theta+\psi)},$$

$M_f(r) = e^{sr^n}$. *Therefore,*

$$\rho = \lim_{r\to+\infty}\frac{\ln\ln M_f(r)}{\ln r} = \lim_{r\to+\infty}\frac{\ln\ln e^{sr^n}}{\ln r} = n.$$

Example 3. *If* $f(z)=\sum_{k=0}^{n} f_k z^k$ *is a polynomial of degree* n, *then* $|f(z)| = |f_n||z|^n(1+o(1))$ *as* $z\to\infty$, *and* $M_f(r) = (1+o(1))|f_n|r^n$ *as* $r\to+\infty$. *Hence,* $\rho = 0$.

Example 4. *If* $f(z) = e^{e^z}$, *then* $M_f(r) = e^{e^r}$. *Thus,* $\rho = +\infty$.

Theorem 1 ([24, 28, 45, 47, 51, 54]). *The order* ρ *of an entire transcendental function* f *is defined by the formulas:*

$$\rho = \overline{\lim_{r\to+\infty}}\frac{\ln\ln\mu_f(r)}{\ln r} = \overline{\lim_{r\to+\infty}}\frac{\ln\nu_f(r)}{\ln r}. \qquad (3)$$

Proof. Indeed, from the inequalities $\mu_f(r)\le M_f(r)\le$ $\le(1+1/\varepsilon)\mu_f\left((1+\varepsilon)r\right)$, where $\varepsilon>0$ and $r\ge 0$, we consequently obtain $\ln\mu_f(r)\le\ln M_f(r)\le(1+o(1))\ln\mu_f\left((1+\varepsilon)r\right)$, $r\to+\infty$, and

$$\frac{\ln\ln\mu_f(r)}{\ln r}\le\frac{\ln\ln M_f(r)}{\ln r}\le$$

$$\le\frac{(1+o(1))\ln\ln\mu_f\left((1+\varepsilon)r\right)}{\ln\left((1+\varepsilon)r\right)}\cdot\frac{\ln\left((1+\varepsilon)r\right)}{\ln r} \text{ as } r\to+\infty.$$

Thus, the first of equalities (3) is proved, and the second follows from the first if we taking into account that

$$\ln\mu_f(r)-\ln\mu_f(r_0)\le\nu_f(r)\int_{r_0}^{r}\frac{dt}{t} = \nu_f(r)\ln\frac{r}{r_0},\ 0<r_0<r, \qquad (4)$$

$$\ln\mu_f(r)-\ln\mu_f(r_0)\ge\int_{r/e}^{r}\frac{\nu_f(t)}{t}dt\ge\nu_f\left(r/e\right),\ r\ge r_0 e. \qquad (5)$$

Theorem 1 is proved. ►

Theorem 2. *For every entire transcendental function* f *holds*

$$\sigma = \overline{\lim_{r\to+\infty}} \frac{\ln \mu_f(r)}{r^\rho}, \tag{6}$$

$$\rho\sigma \le \overline{\lim_{r\to+\infty}} \frac{\nu_f(r)}{r^\rho} \le \sigma e^\rho. \tag{7}$$

Proof. Indeed, the equality (6) is proved in the same way as the first equality in (3). The right-hand side of (7) we obtain from inequality (5). To prove the left-hand side, let us denote the upper bound in (7) by s. Then, if $s<+\infty$, we have $(\forall\varepsilon>0)(\exists t_0)(\forall t\ge t_0): \nu_f(t)\le(s+\varepsilon)t^\rho$. Therefore, as $r\to+\infty$

$$\ln \mu_f(r) \le (s+\varepsilon)\int_1^r t^{\rho-1}dt + O(1) = (s+\varepsilon)(1+o(1))\frac{r^\rho}{\rho}. \ ►$$

Theorem 3 ([24, 28, 45, 47, 51, 54]). *The order* ρ *and the type* σ *of an entire transcendental function* $f(z)=\sum_{k=0}^{\infty} f_k z^k$ *can be defined by formulas (in the first formula, we assume that* $|1/0|=+\infty$ *and* $\ln(+\infty)=+\infty$*):*

$$\rho = \overline{\lim_{k\to+\infty}} \frac{k\ln k}{\ln|1/f_k|}, \tag{8}$$

$$\sigma = \overline{\lim_{k\to+\infty}} \frac{k}{e\rho}|f_k|^{\rho/k}. \tag{9}$$

Proof. Let us denote the right-hand side of (8) by ρ_1. If $\rho_1<+\infty$, then

$$(\forall\varepsilon>0)(\exists c_1)(\forall k\ge1): |f_k|\le c_1\exp\big((k/(\rho_1+\varepsilon))\ln k\big).$$

Therefore,

$$\mu_f(r)\le c_1\exp\left(\frac{t\ln t}{\rho_1+\varepsilon}-t\ln r\right)=c_1\exp\left(\frac{r^{\rho_1+\varepsilon}}{e(\rho_1+\varepsilon)}\right),\ r\ge r_0.$$

Hence,

$$\overline{\lim_{r\to+\infty}} \frac{\ln\ln\mu_f(r)}{\ln r}\le\rho_1,$$

and we have $\rho\le\rho_1$. Prove that $\rho_1\le\rho$. If $\rho=+\infty$, then this inequality is obvious. Assume that $\rho<+\infty$. Then, we have

$$\left(\forall\varepsilon>0\right)\left(\exists c_1\right)\left(\forall r\ge 0\right):M_f\left(r\right)\le c_1\exp\left(r^{\rho+\varepsilon}\right).$$

Therefore, from the Cauchy inequalities, we obtain $|f_k|\le c_1\exp\left(-k\ln r+r^{\rho+\varepsilon}\right)$ for all $r>0$ and $k\ge 0$. Put $r=\left(n/\left(\rho+\varepsilon\right)\right)^{1/(\rho+\varepsilon)}$ (since at this value r, the right-hand side has its minimum). Then

$$\left(\forall k>0\right):|f_k|\le c_1\exp\left(-\frac{k}{\rho+\varepsilon}\ln\frac{k}{\rho+\varepsilon}+\frac{k}{\rho+\varepsilon}\right).$$

This implies that $\rho_1\le\rho$ and the equality (8) is valid. The equality (9) is proved similarly. The theorem is proved. ►

Example 5. *If* $f(z)=\sum_{k=0}^{\infty}\frac{z^k}{(\tau k+1)^{k/s}}$, *where* $\tau\in(0;+\infty)$ *and* $s\in(0;+\infty)$, *then*

$$f_k=\frac{1}{(\tau k+1)^{k/s}}=\frac{1+o(1)}{(\tau k)^{k/s}e^{1/(\tau s)}}$$

as $k\to\infty$. *Hence,* $\rho=s$ *and* $\sigma=1/(\tau es)$.

Example 6. *If* $f(z)=\sum_{n=0}^{\infty}\frac{z^{2n}}{(n+1)^{n/s}}$, *where* $s\in(0;+\infty)$, *then*

$$f_k=\begin{cases}\dfrac{1}{(n+1)^{n/s}}, & k=2n,\\ 0, & k=2n+1,\end{cases}$$

$$f_{2n}=\frac{1}{(n+1)^{n/s}}=\frac{1+o(1)}{n^{n/s}e^{1/s}},\ n\to\infty.$$

Thus, $\rho=2s$ *and* $\sigma=1/(es)$.

Example 7. *If* $f(z)=\sum_{k=0}^{\infty}\frac{z^k}{(k+1)^{k/\ln(2+k)}}$, *then* $\rho=+\infty$.

Example 8. *If* $f(z)=\sum_{k=0}^{\infty}\frac{z^k}{(k+1)^{k\ln k}}$, *then* $\rho=0$.

Example 9. *If* $f(z)=\sum_{k=0}^{\infty}\frac{z^k}{\left((k+1)\ln(2+k)\right)^{k/s}}$, *where* $s\in(0;+\infty)$, *then* $\rho=s$ *and* $\sigma=0$.

Example 10. *If* $f(z)=\sum_{k=0}^{\infty}\frac{z^k}{\left((k+1)/\ln(2+k)\right)^{k/s}}$, *where* $s\in(0;+\infty)$, *then* $\rho=s$ *and* $\sigma=+\infty$.

3.2. The Phragmén-Lindelöf principle. This principle is analogous to the maximum principle for unbounded domains.

Theorem 1 ([24, 28, 45, 47, 51, 52, 54]). *Let the function* f *be holomorphic in the half-plane* $\mathbb{C}_+=\left\{z:\operatorname{Re} z>0\right\}$, *continuous on* $\overline{\mathbb{C}_+}$ *and*

$$(\exists(r_k),0<r_k\uparrow+\infty)(\exists\rho<1)(\exists c_2)(\forall\varphi\in(-\pi/2;\pi/2))(\forall k):$$

$$\left|f(r_ke^{i\varphi})\right|\le c_2\exp(r_k^{\rho}).$$

Then, if $\sup\{\left|f(z)\right|:z\in\partial\mathbb{C}_+\}\le c_1$, *we have* $\sup\{\left|f(z)\right|:z\in\mathbb{C}_+\}\le c_1$.

Proof. Let $\rho<\rho_1<1$, $0<\varepsilon<1$ and $F(z)=f(z)\exp(-\varepsilon z^{\rho_1})$. Let us consider a certain holomorphic branch of this function in $\mathbb{C}_+$ (also denoted by F). Then

$$\sup\left\{\left|F(z)\right|:z\in\partial\mathbb{C}_+\right\}\le c_1,$$

$$\sup\left\{\left|F(r_ke^{i\varphi})\right|:\varphi\in(-\pi/2\,\pi/2)\right\}\le c_2\exp(r_k^{\rho}-\varepsilon r_k^{\rho_1}\cos(\pi\rho_1/2))\to0$$

as $k\to+\infty$, and by applying the maximum principle to semidisks $U_+(0;r_k)=\{z:\left|z\right|<r_k,\operatorname{Re} z>0\}$, for large k we have $\sup\left\{F(z):z\in\mathbb{C}_+\right\}\le c_1$. Hence, $\left|f(z)\right|\le c_1\left|\exp(-\varepsilon\, z^{\rho_1})\right|$ for $z\in\mathbb{C}_+$, and passing to the limit as $\varepsilon\to0$, we obtain the required proposition. Theorem 1 is proved. ►

Corollary 1 ([24, 28, 45, 47, 51, 52, 54]). *Let* $1/2\le\alpha<+\infty$, *and let* f *be a function holomorphic inside an angle* $\mathbb{C}(-\pi/2\alpha;\pi/2\alpha)=\{z:\left|\arg z\right|<\pi/2\alpha\}$ *and continuous on its closure, satisfying*

$$(\exists(r_k),0<r_k\uparrow+\infty)(\exists\rho<\alpha)(\exists c_2)(\forall\varphi,\left|\varphi\right|<\pi/2\alpha)(\forall k):$$

$$\left|f(r_ke^{i\varphi})\right|\le c_2\exp(r_k^{\rho}).$$

Then, if $\sup\left\{\left|f(z)\right|:z\in\partial\mathbb{C}(-\pi/2\alpha;\pi/2\alpha)\right\}\le c_1$, *we have*

$$\sup\{\left|f(z)\right|:z\in C(-\pi/2;\pi/2)\}\le c_1.$$

Proof. To obtain this corollary, it is necessary to consider a holomorphic branch of the function $F(z)=f(z^{1/\alpha})$ in $\mathbb{C}_+$ and apply Theorem 1 to it. ►

Corollary 2. *If an entire function of order less than* 1 *is bounded on the real axis, then this function is constant.*

Corollary 3. *If an entire function of order less than* $1/2$ *is bounded on the positive real axis, then this function is constant.*

Corollary 4. *If an entire function of order less than* ρ *is bounded on the sides of an angle of opening* π/ρ*, then it is bounded inside this angle.*

Example 1. *An entire function* $f(z)=e^z$ *has the order* $\rho=1$ *and is bounded on the imaginary axis.*

Example 2. *An entire function* $f(z)=\cos\sqrt{z}$ *has the order* $\rho=1/2$ *and is bounded on the positive real semiaxis.*

3.3. ρ**-trigonometrically convex functions.** Let $0<\rho<+\infty$. A function $h:[\alpha;\beta]\to[-\infty;+\infty)$ is called ρ-trigonometrically convex on the closed segment $[\alpha;\beta]$ if for any θ, θ_1 and θ_2, $\alpha\le\theta_1<\theta<\theta_2\le\beta$, $\theta_2-\theta_1<\pi/\rho$, holds [13, 24, 28, 45, 47, 51, 54]:

$$h(\theta)\le h(\theta_1)\frac{\sin\rho(\theta_2-\theta)}{\sin\rho(\theta_2-\theta_1)}+h(\theta_2)\frac{\sin\rho(\theta-\theta_1)}{\sin\rho(\theta_2-\theta_1)}\,. \tag{1}$$

This inequality can be rewritten as:

$$h(\theta)\sin\rho(\theta_2-\theta_1)+h(\theta_1)\sin\rho(\theta-\theta_2)+h(\theta_2)\sin\rho(\theta_1-\theta)\le 0\,, \tag{2}$$

or in this form:

$$h(\theta)\le\frac{h(\theta_1)\sin\rho\theta_2-h(\theta_2)\sin\rho\theta_1}{\sin\rho(\theta_2-\theta_1)}\cos\rho\theta+$$

$$+\frac{h(\theta_2)\cos\rho\theta_1-h(\theta_1)\cos\rho\theta_2}{\sin\rho(\theta_2-\theta_1)}\sin\rho\theta\,. \tag{3}$$

A function $h:(\alpha;\beta)\to[-\infty;+\infty)$ is called ρ-trigonometrically convex on the open segment $(\alpha;\beta)$ if it is ρ-trigonometrically convex on every closed segment $[a;d]\subset(\alpha;\beta)$.

Theorem 1. *If a function* $h\ne-\infty$ *is* ρ*-trigonometrically convex on* $(\alpha;\beta)$ *then* $h(\alpha;\beta)\subset(-\infty;+\infty)$.

Proof. Assume that there exists a point $\theta_1\in(\alpha;\beta)$ such that $h(\theta_1)\ne-\infty$. Since $h\ne-\infty$, a value θ_1 can be chosen such that there exists $\theta_2\in(\alpha;\beta)$ satisfying $h(\theta_2)\ne-\infty$ for $0<\theta_2-\theta_1<\pi/\rho$. Then, from (1), we obtain $h(\theta)=-\infty$ for $\theta\in(\theta_1;\theta_2)$. Further, by putting $\theta=\theta_2,\ \theta=\theta_1,\theta_2=\theta_3\in(0;\beta),\theta_3-\theta_2<\pi/\rho$ in (1), we get $h(\theta_2)=-\infty$. This contradiction proves the theorem. ►

Theorem 2. *Let* $h \neq -\infty$ *be a* 2π*-periodic and* ρ*-trigonometrically convex function on* $(-\infty;+\infty)$*. Then*

$$\left(\forall \theta_1 \in \mathbb{R}\right):\, h(\theta_1) + h(\theta_1 + \pi/\rho) \geq 0\,, \tag{4}$$

and

$$\min\left\{h(\theta): \theta \in (-\infty;+\infty)\right\} \geq -\max\left\{h(\theta): \theta \in (-\infty;+\infty)\right\}. \tag{5}$$

Proof. To obtain (4), it is sufficient sending θ_2 to $\theta_1 + \pi/\rho$ in inequality (2). The inequality (5) follows as a consequence of (4). Teorem 2 is proved. ►

Theorem 3 ([13, 24, 28, 45, 47, 51, 54]). *If a* ρ*-trigonometrically convex function* $h \neq -\infty$ *in an interval* $(\alpha;\beta)$ *is bounded, i.e.,* $|h(\theta)| < K$ *for* $\theta \in (\alpha;\beta)$*, then it is a continuous function of* $\theta \in (\alpha;\beta)$*, and it satisfies a Lipschitz condition:*

$$\left(\exists c_1 > 0\right)\left(\forall (t;\tau) \in (\alpha;\beta)\times(\alpha;\beta)\right): \left|h(t) - h(\tau)\right| \leq c_1 \left|t - \tau\right|.$$

Theorem 4 ([13, 24, 28, 45, 47, 51, 54]). *Let* $h \neq -\infty$ *be a* ρ*-trigonometrically convex function in an interval* $(\alpha;\beta)$*. Then it has the right hand* $h'_+(\varphi)$ *and left hand* $h'_-(\varphi)$ *derivatives at every point* $\varphi \in (\alpha;\beta)$*. Moreover, a)* $h'_+(\varphi) = h'_+(\varphi+0)$*; b)* $h'_-(\varphi) = h'_-(\varphi-0)$*; c)* $h'_-(\varphi) \leq h'_+(\varphi)$*.*

For a twice continuously differentiable and 2π-periodic function h to be a ρ-trigonometrically convex, it is necessary and sufficient that $h''(\theta) + \rho^2 h(\theta) \geq 0$ for $\theta \in \mathbb{R}$.

Theorem 5 ([13, 24, 28, 45, 47, 51, 54]). *In order that a* 2π*-periodic function* $h: \mathbb{R} \to \mathbb{R}$ *be a* ρ*-trigonometrically convex function on* $[0;2\pi]$*, it is necessary and sufficient that it can be represented in the form*

$$h(\theta) = \frac{1}{2\rho \sin \pi\rho} \int_0^{2\pi} \cos \rho\left(\left|\theta - t\right| - \pi\right) ds(t)\,,\ \theta \in [0;2\pi]\,,$$

for noninteger ρ*, where* s *is a nondecresing and continuous from the left on the segment* $[0;2\pi]$ *function, and for integer* ρ *in the form*

$$h(\theta) = \frac{1}{2\pi\rho} \int_{\theta-2\pi}^{\theta} \left(\theta - t\right) \sin \rho\left(t - \theta\right) ds(t) + c_\rho e^{i\rho\theta} + \overline{c_\rho} e^{-i\rho\theta}\,,\ \theta \in [0;2\pi]\,,$$

where c_ρ *is constant and* s *is a nondecresing and continuous from the left on* $[0;2\pi]$ *function satisfying the condition*

$$\int_0^{2\pi} e^{i\rho x} ds(x) = 0 .$$

Example 1. *A function* $h(\theta) = \cos\theta$ *is* 1*-trigonometrically convex on* $(-\infty;+\infty)$, *because* $h''(\theta) + \rho^2 h(\theta) = -\cos\theta + \cos\theta = 0$.

3.4. The indicator function. The indicator of a function $f : \mathbb{C} \to \mathbb{C}$ is called a function $h(\theta) = h_f(\theta) = h(\theta; f) = h_\rho(\theta; f)$ defined by [13, 24, 28, 45, 47, 51, 54]:

$$h(\theta) = \overline{\lim_{r\to+\infty}} \frac{\ln\left|f(re^{i\theta})\right|}{r^\rho}, \tag{1}$$

where $\rho \in (0;+\infty)$ is a given number. If f is an entire function, then by ρ take usually its order and sometimes even its formal order.

Theorem 1 ([13, 24, 28, 45, 47, 51, 54]). *Let* f *be an entire function of normal type with respect to the order* $\rho \in (0;+\infty)$. *Then the fundamental relation for the indicator function holds*

$$h(\theta) \le H(\theta;\theta_1;h(\theta_1);\theta_2;h(\theta_2)), \tag{2}$$

for all values θ_1, θ_2 *and* θ *in* $[0;2\pi]$ *such that* $\theta_2 - \theta_1 < \pi/\rho$, *where*

$$H(\theta;\theta_1;h_1;\theta_2;h_2) =$$

$$= \frac{h_1 \sin\rho\theta_2 - h_2 \sin\rho\theta_1}{\sin\rho(\theta_2-\theta_1)}\cos\rho\theta + \frac{h_2\cos\rho\theta_1 - h_1\cos\rho\theta_2}{\sin\rho(\theta_2-\theta_1)}\sin\rho\theta .$$

Proof. Let us consider an arbitrary holomorphic branch of the function $\omega(z) = \exp((a-bi)z^\rho)$ in an angle $\mathbb{C}(0;2\pi) = \{z : 0 < \arg z < 2\pi\}$. Then $h(\theta;\omega) = a\cos\rho\theta + b\sin\rho\theta$. We choose a and b such that $h(\theta_1;\omega) = h_1$ and $h(\theta_2;\omega) = h_2$, where $h_1 = h(\theta_1; f) + \delta$, $h_2 = h(\theta_2; f) + \delta$ and $\delta > 0$, by solving the system:

$$\begin{cases} a\cos\rho\theta_1 + b\sin\rho\theta_1 = h_1, \\ a\cos\rho\theta_2 + b\sin\rho\theta_2 = h_2. \end{cases}$$

We find that

$$a = \frac{h_1\sin\rho\theta_2 - h_2\sin\rho\theta_1}{\sin\rho(\theta_2-\theta_1)}, \qquad b = \frac{h_2\cos\rho\theta_1 - h_1\cos\rho\theta_2}{\sin\rho(\theta_2-\theta_1)}.$$

Thus, for such constants a and b, we have $h(\theta;\omega) = H(\theta,\theta_1,h_1,\theta_2,h_2)$. Let $\psi(z) = f(z)\exp(-(a-bi)z^\rho)$. Then

$$h(\theta;\psi) = h(\theta; f) - h(\theta;\omega), \; h(\theta_1;\psi) = h(\theta_2;\psi) = -\delta .$$

By the Phragmén-Lindelöf principle, the function ψ is bounded inside an angle $\mathbb{C}(\theta_1;\theta_2)$. Hence, $h(\theta;\psi)\le 0, \theta\in[\theta_1;\theta_2]$. From this, due to the arbitrariness of the choice of $\delta>0$, we obtain the required statement. Theorem 1 is proved. ►

Corollary 1. *If f is an entire function of normal type with respect to the order $\rho\in(0;+\infty)$, then its indicator function h is a continuous, 2π-periodic ρ-trigonometrically convex function on $\mathbb{R}$.*

From the definition of the indicator function, it follows that for an entire function f of normal type with respect to the order $\rho\in(0;+\infty)$, the following holds [13, 24, 28, 45, 47, 51, 54]:

$$(\forall\theta\in[0;2\pi])(\forall\varepsilon>0)(\exists c_1)(\forall r\ge 0):\left|f(re^{i\varphi})\right|\le c_1\exp((h(\theta)+\varepsilon)r^{\rho}).$$

Along with this, the following theorem by S. Bernstein holds true:

Theorem 2 ([13, 24, 28, 45, 47, 51, 54]). *If f is an entire function of normal type with respect to the order $\rho\in(0;+\infty)$, then*

$$(\forall\varepsilon>0)(\exists c_1)(\forall\theta\in[0;2\pi])(\forall r\ge 0):\left|f(re^{i\varphi})\right|\le c_1\exp((h(\theta)+\varepsilon)r^{\rho}).$$

Corollary 2. *If f is an entire function of normal type σ with respect to the order ρ, then* $\max\{h(\theta):\theta\in[0;2\pi]\}=\sigma$.

Corollary 3. *For every entire function f of normal type σ with respect to the order ρ holds $-\sigma\le h(\theta)\le\sigma$ for $\theta\in[0;2\pi]$.*

Example 1. *If* $f(z)=e^z$, *then* $\left|f(re^{i\theta})\right|=e^{r\cos\theta}$, $M_f(r)=e^r$, $\rho=\lim\limits_{r\to+\infty}\dfrac{\ln\ln M_f(r)}{\ln r}=1$ *and*

$$h_f(\theta)=\varlimsup_{r\to+\infty}\frac{\ln\left|f(re^{i\theta})\right|}{r^{\rho}}=\varlimsup_{r\to+\infty}\frac{\ln e^{r\cos\theta}}{r}=\cos\theta.$$

Example 2. *If* $f(z)=e^{\tau z^n}$, *where* $\tau=se^{i\psi}\in\mathbb{C}$ *and* $n\in\mathbb{N}$, *then* $\left|f(re^{i\theta})\right|=e^{sr^n\cos(n\theta+\psi)}$, $M_f(r)=e^{sr^n}$, $\rho=\lim\limits_{r\to+\infty}\dfrac{\ln\ln M_f(r)}{\ln r}=n$ *and*

$$h_f(\theta)=\varlimsup_{r\to+\infty}\frac{\ln\left|f(re^{i\theta})\right|}{r^{\rho}}=\varlimsup_{r\to+\infty}\frac{\ln e^{sr^n\cos(n\theta+\psi)}}{r^n}=s\cos(n\theta+\psi).$$

Example 3. *Let* $f(z)=\sin z$. *Then* $\rho=1$ *and* $h_f(\theta)=|\sin\theta|$.

Example 4. *Let* $f(z)=\cos z$. *Then* $\rho=1$ *and* $h_f(\theta)=|\sin\theta|$.

3.5. Supporting functions of convex sets. A set D is called convex if, together with any two of its points, it also contains the line segment connecting them. The convex hull of a set $D \subset \mathbb{C}$ is the set $\operatorname{conv} D$ that is the intersection of all closed convex sets containing it [24, 45, 47, 51, 54]. Alternatively, the convex hull of a set D can be described as the smallest closed convex set that contains it. It is the intersection of all half-planes containing the set D. If a set D is convex and closed, then $\operatorname{conv} D = D$.

The supporting function of a set $D \subset \mathbb{C}$ is called the function [24, 45, 47, 51, 54]:

$$k_D(\theta) = \sup\{\operatorname{Re}(ze^{-i\theta}) : z \in D\}. \qquad (1)$$

The line $x\cos\theta + y\sin\theta = k_D(\theta)$ is called a supporting line of a set D. For each $\theta \in [0; 2\pi]$ the set D lies in a half-plane $\{z : x\cos\theta + y\sin\theta \le k_D(\theta)\}$.

Example 1. *If* $D = \left\{a = |a|e^{i\psi}\right\}$, *then* $\operatorname{conv} D = D$ *and*

$$k_D(\theta) = |a|\cos(\psi - \theta).$$

Example 2. *If* $\sigma > 0$ *and* $D = \{z : -\sigma \le \operatorname{Re} z \le \sigma, \operatorname{Im} z = 0\}$, *then* $\operatorname{conv} D = D$ *and* $k_D(\theta) = \max\{x\cos\theta : x \in [-\sigma; \sigma]\} = \sigma|\cos\theta|$.

Example 3. *Let* $\sigma > 0$ *and* $D = \{-\sigma; \sigma\}$. *Then*

$$\operatorname{conv} D = \{z : -\sigma \le \operatorname{Re} z \le \sigma, \operatorname{Im} z = 0\}$$

and $k_D(\theta) = \max\{x\cos\theta : x \in \{-\sigma; \sigma\}\} = \sigma|\cos\theta|$.

Example 4. *Let* $\sigma > 0$ *and* $D = \{-\sigma; \sigma; i\sigma; -i\sigma\}$. *Then*

$$\operatorname{conv} D = \{z : \operatorname{Im} z \le \sigma - \operatorname{Re} z\} \cap \{z : \operatorname{Im} z \le \sigma + \operatorname{Re} z\} \cap$$
$$\cap \{z : \operatorname{Im} z \ge -\sigma - \operatorname{Re} z\} \cap \{z : \operatorname{Im} z \ge -\sigma + \operatorname{Re} z\}$$

and

$$k_D(\theta) = \max\{x\cos\theta + y\sin\theta : z = x + iy \in \{-\sigma; \sigma; i\sigma; -i\sigma\}\} =$$

$$= \begin{cases} \sigma\cos\theta, & \theta \in [-\pi/4; \pi/4], \\ \sigma\sin\theta, & \theta \in [\pi/4; 3\pi/4], \\ -\sigma\cos\theta, & \theta \in [3\pi/4; 5\pi/4], \\ -\sigma\sin\theta, & \theta \in [5\pi/4; 7\pi/4], \end{cases} = \begin{cases} \sigma\cos\theta, & \theta \in [0; \pi/4], \\ \sigma\sin\theta, & \theta \in [\pi/4; 3\pi/4], \\ -\sigma\cos\theta, & \theta \in [3\pi/4; 5\pi/4], \\ -\sigma\sin\theta, & \theta \in [5\pi/4; 7\pi/4], \\ \sigma\cos\theta, & \theta \in [7\pi/4; 2\pi]. \end{cases}$$

3.6. The Borel transform. Space PW_σ^2. An entire function L is called a function of exponential type $\le \sigma$ if [24, 45, 47, 51, 54]

$$(\forall \varepsilon > 0)(\exists c_1 > 0)(\forall z \in \mathbb{C}) : |L(z)| \le c_1 \exp\big((\sigma + \varepsilon)|z|\big). \qquad (1)$$

For every entire function L of exponential type $\le \sigma$, we have:

$$\overline{\lim_{r\to\infty}}\frac{\ln M_L(r)}{r}=\overline{\lim_{n\to+\infty}}\frac{n}{e}|L_n|^{1/n}=\overline{\lim_{n\to\infty}}|n!L_n|^{1/n}\le\sigma\ ,\ L(z)=\prod_{n=0}^{\infty}L_n z^n\ . \qquad (2)$$

Therefore, the order of an entire function of exponential type is less than 1, provided that in this definition, condition (1) can be replaced by condition (2).

The function

$$\gamma_L(z)=\sum_{n=0}^{\infty}\frac{n!L_n}{z^{n+1}}$$

is called [24, 45, 47, 51, 54] the Borel transform of the function L. If the function L satisfies condition (1), then the last series converges for $|z|>\sigma$ and γ_L is a holomorphic function in the domain $\{z:|z|>\sigma\}$. The convex hull G_L of a set of finite singular points of the function γ_L is called [24, 45, 47, 51, 54] the conjugate indicator diagram of the function L.

Let

$$h_L(\theta)=\overline{\lim_{r\to+\infty}}\frac{\ln|L(re^{i\theta})|}{r},$$

be the indicator function of an entire function L with respect to the formal order $\rho=1$ and G_L is conjugate indicator diagram of L.

Theorem 1 ([24, 45, 47, 51, 54]). *If L is an entire function of exponential type, then for all $\theta\in[0;2\pi]$ holds $h_L(\theta)\le k_{G_L}(-\theta)$ and*

$$L(z)=\frac{1}{2\pi i}\int_{\Gamma}\gamma(t)e^{tz}dt\ ,\ z\in\mathbb{C}, \qquad (3)$$

where Γ is a closed rectifiable Jordan curve such that $G_L\subset\operatorname{Int}\Gamma$.

Proof. Indeed,

$$\frac{1}{2\pi i}\int_{\Gamma}\gamma(t)e^{tz}dt=\sum_{n=0}^{\infty}\frac{L_n n!}{2\pi i}\int_{\Gamma}\frac{e^{tz}dt}{t^{n+1}}=\sum_{n=0}^{\infty}L_n z^n=L(z),$$

which implies (3). Further, let $G(\varepsilon)=\{z+\varepsilon e^{i\psi}:\psi\in[0;2\pi],z\in G\}$. Then $k_{G(\varepsilon)}(\theta)\le k_G(\theta)+\varepsilon$. Therefore, by using (3), we obtain

$$|L(re^{i\varphi})|=\left|\frac{1}{2\pi i}\int_{\partial G(\varepsilon)}\gamma_L(t)e^{tre^{i\varphi}}dt\right|\le\exp(rk_{G(\varepsilon)}(-\varphi))\frac{1}{2\pi i}\int_{\partial G(\varepsilon)}|\gamma(t)||dt|.$$

which proves $h_L(\varphi)\le k_G(-\varphi)$. Theorem 1 is proved. ►

Corollary 1. *If L is an entire function of exponential type, then*

$$(\forall\varepsilon>0)(\exists c_1>0)(\forall z=re^{i\varphi}\in\mathbb{C}):|L(z)|\le c_1\exp(r(k_G(-\varphi)+\varepsilon)).$$

Theorem 2 ([24, 45, 47, 51, 54]). *Let* L *be an entire function of exponential type. Then for each* $\theta\in[0;2\pi]$ *the function* γ *is holomorphic in the half-plane* $\mathbb{C}_\theta=\{z:\mathrm{Re}(ze^{i\theta})>h_L(\theta)\}$ *and inside this half-plane, we have:*

$$\gamma_L(z)=\int_0^{+\infty e^{i\theta}} L(\varsigma)e^{-\varsigma z}d\varsigma\,. \tag{4}$$

Proof. Since $\left|L(\rho e^{i\theta})\right|\le\exp((h_L(\theta)+o(1))\rho),\ \rho\to+\infty$, the integral (4) uniformly converges on the compact sets in C_θ. Further,

$$L(z)=\sum_{k=0}^{n}L_kt^k+r_n(t),\quad r_n(t):=\sum_{k=n+1}^{\infty}L_kt^k$$

and

$$\int_0^{\infty e^{i\theta}}\left(\sum_{k=0}^{n}L_kt^k\right)e^{-tz}dt=\sum_{k=0}^{n}L_k\int_0^{\infty e^{i\theta}}t^ke^{-tz}dt=\sum_{k=0}^{n}\frac{L_kk!}{z^{k+1}}\,.$$

Furthermore,

$$\int_{\tau e^{i\theta}}^{\infty e^{i\theta}} r_n(t)e^{-zt}dt\to0,\ \tau\to+\infty,\ \mathrm{Re}(ze^{i\theta})\ge\sigma+3,$$

and for every $\tau>0$

$$\int_0^{\tau e^{i\theta}} r_n(t)e^{-zt}dt\to0,\ n\to\infty,\ z\in\mathbb{C},$$

because

$$(\forall\varepsilon>0)(\exists c_1>0)(\forall n\ge1):\left|L_n\right|\le c_1((\sigma+\varepsilon)e/n)^n,$$

$$(\forall\varepsilon>0)(\exists c_2>0)(\forall t\in C)(\forall n):\left|r_n(t)\right|\le c_1\sum_{k=n+1}^{\infty}\left(\frac{(\sigma+\varepsilon)e}{n}\right)^n\left|t\right|^n\le$$

$$\le c_2e^{(\sigma+2\varepsilon)|t|}\sum_{k=n+1}^{\infty}\left(\frac{\sigma+\varepsilon}{\sigma+2\varepsilon}\right)^k.$$

Therefore, if $\mathrm{Re}(ze^{i\theta})\ge\sigma+3$, then

$$\left|r_n(\rho e^{i\theta})e^{-z\rho e^{i\theta}}\right|\le\varepsilon_ne^{(\sigma+2\varepsilon)\rho-\rho\mathrm{Re}(ze^{i\theta})}\le\varepsilon_ne^{-\rho},$$

where $\varepsilon_n\to0$ as $n\to\infty$. Hence, the equality (4) holds for $\mathrm{Re}(ze^{i\theta})\ge\sigma+3$. Finally, it remains to remark that $h_L(\theta)\le\sigma$ and the integral (4) uniformly converges on the compact sets in C_θ. Theorem 2 is proved. ►

Theorem 3 (Pólya) ([24, 45, 47, 51, 54]). *For every entire function* L *of exponential type the relation* $h_L(\theta)=k_{G_L}(-\theta)$ *holds for* $\theta\in[0;2\pi]$.

Proof. The function γ is holomorphic in $\mathbb{C}/G$. For every $\theta\in[0;2\pi]$ the set G lies in a half-plane $\{z:\mathrm{Re}(ze^{i\theta})\le k_G(-\theta)\}$. Therefore, γ is holomorphic in $\{z:\mathrm{Re}(ze^{i\theta})>k_G(-\theta)\}$ and is not holomorphic in any other half-plane that contains it. Hence, from Theorem 2 it follows that $k_G(-\theta)\le h_L(\theta)$. Moreover, by Theorem 1 we have $h_L(\theta)\le k_G(-\theta)$ and this proves the theorem. ►

The set $G_L^{\bullet}$ satisfying $k_{G_L^{\bullet}}(\theta)=h(\theta)$ is called [24, 45, 47, 51, 54] the indicator diagram of an entire function L of exponential type. The conjugate indicator diagram of every entire function of exponential type is obtained from its indicator diagram by reflecting it symmetrically with respect to the real axis.

Example 1. *If* $a=|a|e^{i\psi}$ *and* $L(z)=e^{az}$, *then*

$$\gamma_L(z)=\sum_{k=0}^{\infty}\frac{a^k}{z^{k+1}}=\frac{1}{z-a},\ G_L=\{a\},\ G_L^{\bullet}=\{\bar{a}\},\ h_L(\theta)=|a|\cos(\psi+\theta),$$

$k_{G_L}(\theta)=|a|\cos(\psi-\theta)$ *and* $k_{G_L^{\bullet}}(\theta)=|a|\cos(\psi+\theta)=h_L(\theta)$.

Example 2. *If* $a>0$ *and* $L(z)=\mathrm{ch}(az)$, *then*

$$\gamma_L(z)=\sum_{k=0}^{\infty}\left(\frac{a^k}{2z^{k+1}}+\frac{(-a)^k}{2z^{k+1}}\right)=\sum_{k=0}^{\infty}\left(\frac{a^k}{2z^{k+1}}+\frac{(-a)^k}{2z^{k+1}}\right)=\frac{1}{2}\left(\frac{1}{z-a}+\frac{1}{z+a}\right),$$

$G_L=[-a;a]=G_L^{\bullet}$ *and* $k_{G_L}(\theta)=a|\cos\theta|=k_{G_L^{\bullet}}(\theta)=h(\theta)$.

Example 3. *If* $\lambda_k\in\mathbb{C}$ *and* $L(z)=\sum_{k=1}^{n}e^{\lambda_k z}$, *then*

$$\gamma_L(z)=\sum_{k=1}^{n}\frac{1}{z-\lambda_k}.$$

Denote by PW_σ^2 the set of all entire functions of exponential type $\le\sigma\in(0;+\infty)$ whose narrowing on $\mathbb{R}$ belongs to the space $L_2(\mathbb{R})$.

Theorem 4 (Paley-Wiener) ([47, 52, 54]). *The class* PW_σ^2 *is composed of functions* G *representable in the form*

$$G(z)=\frac{1}{\sqrt{2\pi}}\int_{-\sigma}^{\sigma}e^{itz}\,\mathrm{g}(t)dt,\ g\in L_2(-\sigma;\sigma).$$

Moreover, $\|g\|_{L_2(-\sigma;\sigma)}=\|G\|_{L_2(\mathbb{R})}$.

Example 4. *The function* $G(z) = z^{-1}\sin(\pi z)$ *belongs to* PW_{π}^{2} .

Remark that, an aditional properties of the indicator function, Borel transform and other variants of the Phragmén-Lindelöf principle can be found in [13, 24, 28, 30-34, 45-47, 49, 51, 52, 54].

3.7. Self-control questions.

1. Formulate the definition of the order of an entire function.
2. Formulate the definition of the type of an entire function.
3. Formulate and prove the theorem on the relationship between the order and type of an entire function and its Taylor coefficients.
4. Formulate and prove the Phragmén-Lindelöf principle for a half-plane.
5. Formulate and prove the Phragmén-Lindelöf principle for an angle.
6. Formulate the definition of a ρ-trigonometrically convex function.
7. Formulate the definition of the indicator function of an entire function.
8. Formulate and prove the theorem on the fundamental relation for the indicator function.
9. Formulate the definition of an entire function of exponential type.
10. Formulate the definition of a convex set.
11. Formulate the definition of the supporting function of a set.
12. Formulate the definition of the convex hull of a set.
13. Formulate the definition of the Borel transform of an entire function.
14. Formulate the definition of the indicator diagram of an entire function.
15. Formulate the definition of the conjugate indicator diagram of an entire function.
16. Formulate and prove Pólya's theorem.
17. Formulate the definition of the space PW_{σ}^{2} .
18. Formulate the Paley-Wiener theorem.

3.8. Exercises and problems.

1. Find the order and type of an entire function f :

1. $f(z) = 1 + z^2$.
2. $f(z) = e^{-z^2}$.
3. $f(z) = e^{-2z^3}$.
4. $f(z) = 1 - 3z^3$.
5. $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^{2n}}$.
6. $f(z) = \sum_{n=1}^{\infty} \frac{z^{2n}}{n^n}$.
7. $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)}$.
8. $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^{n^{1+\alpha}}}$, $\alpha > 0$.

2. Find the indicator function of an entire function f :

1. $f(z) = e^{-(1-i)z}$.
2. $f(z) = e^{(1+i)z}$.

3. $f(z)=e^{3z^2}$.

4. $f(z)=e^{2iz^3}$.

5. $f(z)=e^{2iz}+z$.

6. $f(z)=ze^{iz}$.

7. $f(z)=e^{z}+z^2$.

8. $f(z)=e^{-z^n}$, $n\in\mathbb{N}$.

3. Let f and g be entire functions. Prove the following statements:

1) $\rho[f\cdot g]\le\rho[f]\cdot\rho[g]$. Using the example of functions $f(z)=e^{-z^2}$ and $g(z)=e^{z^2}$, verify that a strict inequality is possible in this case;

2) $\rho[f+g]\le\rho[f]+\rho[g]$. Using the example of functions $f(z)=1-e^{z}$ and $g(z)=e^{z}$, verify that a strict inequality is possible in this case;

3) $\rho[f/g]\le\rho[f]+\rho[g]$ if f/g is an entire function. Using the example of functions $f(z)=e^{z}$ and $g(z)=e^{z}$, verify that a strict inequality is possible in this case.

4. Let f and g be entire functions of order $\rho\in(0;+\infty)$ with indicators h_f and h_g, respectively. Prove the following statements:

1) $h_{f\cdot g}(\varphi)\le h_f(\varphi)+h_g(\varphi)$, $\varphi\in[0;2\pi]$;

2) $h_{f+g}(\varphi)\le\max\left\{h_f(\varphi);h_g(\varphi)\right\}$, $\varphi\in[0;2\pi]$.

5. Prove the following statements (see [13, 24, 26, 28, 31-34, 45-47, 51, 52, 54]):

1. If the function f is holomorphic in $\mathbb{C}_+$ and continuous and bounded in $\overline{\mathbb{C}}_+$, then

$$f(z)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{x\,f(it)dt}{(y-t)^2+x^2},\ z=x+iy\in\mathbb{C}_+.$$

2. If the function f is holomorphic in $\mathbb{C}_+$, and continuous and bounded in $\overline{\mathbb{C}}_+$ such that $\sup\{|f(z)|:z\in\partial\mathbb{C}_+\}\le c_1<+\infty$, then $\sup\{|f(z)|:z\in\mathbb{C}_+\}\le c_1$.

3. If the function $\eta:(-\infty;+\infty)\to[0;+\infty)$ is an even and nondecreasing on $[0;+\infty)$ satisfying $\int_1^{+\infty}t^{-2}\eta(t)dt<+\infty$, then the function

$$U(z)=\frac{2(x+1)}{\pi}\int_{-\infty}^{+\infty}\frac{\eta(t)dt}{(y-t)^2+x^2}$$

is harmonic in a half-plane $\{z:\operatorname{Re}z>-1\}$ and $U(z)\ge\eta(|z|)$, $z=x+iy\in\overline{\mathbb{C}}_+$.

4. Let the function η satisfies the conditions of the previous statement. Then there exists a function G holomorphic in $\overline{\mathbb{C}}_+$ such that

$$(\forall z \in \overline{\mathbb{C}}_+): |G(z)| \le \exp(-\eta(z)).$$

5. Let the function f is holomorphic in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}}_+$, let $\overline{\lim\limits_{x\to+\infty}} x^{-1}\ln|f(x)| \le 0$, $\sup\{|f(z)| : z \in \partial\mathbb{C}_+\} \le c_1$ and there is $\beta \in (0;2)$ such that the condition $\sup\{|f(z)|\exp(-\varepsilon|z|^{\beta}) : z \in \mathbb{C}_+\} < +\infty$ holds for all $\varepsilon > 0$. Then $\sup\{f(z) : z \in \mathbb{C}_+\} \le c_1$, where $c_1 > 0$.

6. Let the function f is holomorphic in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}}_+$, for which $(\exists c_2 > 0)(\forall z \in \mathbb{C}_+) : |f(z)| \le c_2 e^{c_2|z|}$. Then, if $\sup\{|f(z)| : z \in \partial\mathbb{C}_+\} \le c_1$ for some $c_1 > 0$ and $\overline{\lim\limits_{x\to+\infty}} x^{-1}\ln|f(x)| \le 0$, we have $|f(z)| \le c_1 \exp(a \operatorname{Re} z)$ for $z \in \mathbb{C}_+$.

7. A ρ-trigonometrically convex function $h \ne -\infty$ on $(\alpha;\beta)$ has both right $h'_+(\theta)$ and left $h'_-(\theta)$ derivatives at every point $\theta \in (\alpha,\beta)$. In addition, $h'_+(\theta) \ge h'_-(\theta)$, while the right derivative h'_+ is continuous from the right and the left derivative h'_- is continuous from the left. Moreover, we have

$$h'_-(\theta_2) - h'_+(\theta_1) + \rho^2 \int_{\theta_1}^{\theta_2} h(t)dt \ge 0, \ \alpha < \theta_1 < \theta_2 < \beta.$$

8. A ρ-trigonometrically convex function $h \ne -\infty$ on $(\alpha;\beta)$ has a derivative everywhere except possibly at a countable number of points. Moreover, if $\theta_0 \in (\alpha;\beta)$ is a point of maximum, then there exists the derivative $h'(\theta_0)$ and $h'(\theta_0) = 0$. In the case of a minimum point, demonstrate using the example of a function $h(\theta) = |\sin\theta|$ that the derivative $h'(\theta_0)$ may not exist, however, $h'(\theta_0) \le 0 \le h'_+(\theta)$.

9. If $h \ne -\infty$ is a ρ-trigonometrically convex function on $[0;2\pi]$ and 2π-periodic function, and θ_0 is its maximum point, then for every θ such that $|\theta - \theta_0| \le \pi/\rho$ holds $h(\theta) \ge h(\theta_0)\cos\rho(\theta - \theta_0)$.

6. Find the convex hull $\operatorname{conv} D$ of a set $D \subset \mathbb{C}$:

1. $D = \{z_0\}$, $z_0 \in \mathbb{C}$.
2. $D = \{z_1; z_2\}$, $z_1, z_2 \in \mathbb{C}$.

3. $D=\left\{z_1;z_2;\ldots;z_{n_0}\right\}$, $z_j\in\mathbb{C}$, $j\in\{1;2;\ldots;n_0\}$.

7. Find the supporting function $k_D(\theta)$ of a set $D\subset\mathbb{C}$:

1. $D=\left\{z_0\right\}$, $z_0\in\mathbb{C}$.
2. $D=\left\{z\in\mathbb{C}:|z|\le R_0\right\}$, $R_0>0$.
3. $D=\left\{z\in\mathbb{C}:z=iy,|y|\le\sigma\right\}$, $\sigma>0$.
4. $D=\left\{-i\sigma;i\sigma\right\}$, $\sigma\in\mathbb{R}\setminus\{0\}$.
5. $D=[1;3]$.
6. $D=\left\{z\in\mathbb{C}:|\operatorname{Re}z|<\pi,\operatorname{Im}z>0\right\}$.

8. Find the Borel transform of the function L:

1. $L(z)=e^{-iz}$.
2. $L(z)=e^{-iz}-e^{z}$.
3. $L(z)=i\operatorname{ch}z$.
4. $L(z)=i\operatorname{sh}z$.
5. $L(z)=z+3z^2$.
6. $L(z)=2z+e^{-z}$.
7. $L(z)=\operatorname{ch}(iz)$.
8. $L(z)=e^{iz}+e^{z}+e^{-iz}$.

3.9. Individual tasks.

1. Find the order and type of an entire function f:

1. $f(z)=1+3z^2$.
2. $f(z)=e^{z^2}$.
3. $f(z)=z^2e^{3z}$.
4. $f(z)=z^2e^{2z}-e^{3z}$.
5. $f(z)=e^{-3z^2}+z^2$.
6. $f(z)=e^{z^2}+1+z$.
7. $f(z)=e^{e^z}$.
8. $f(z)=\operatorname{ch}z$.
9. $f(z)=e^{5z}-3e^{2z^3}$.
10. $f(z)=\sin z$.
11. $f(z)=e^{(2-i)z^2}$.
12. $f(z)=e^{(1+i)z}$.
13. $f(z)=\sum\limits_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{(2n+1)!}z^n$.
14. $f(z)=\sum\limits_{n=2}^{\infty}\left(\frac{\ln n}{n}\right)^n z^n$.
15. $f(z)=e^{z}+e^{iz}$.
16. $f(z)=\operatorname{sh}z$.
17. $f(z)=\sum\limits_{k=0}^{\infty}\frac{z^k}{(k+1)^{2k\ln k}}$.
18. $f(z)=\sum\limits_{k=0}^{\infty}\frac{2^k}{(k!)^3}z^k$.
19. $f(z)=\sum\limits_{k=0}^{\infty}\frac{4^k}{\left((k+1)^2\ln(2+k)\right)^{k/3}}z^k$.
20. $f(z)=\sum\limits_{k=0}^{\infty}\frac{4^k}{(k!)^3}z^{2k}$.

21. $f(z)=\sum_{k=0}^{\infty}\frac{z^k}{(4k)!}$.

22. $f(z)=\sum_{n=1}^{\infty}\left(\frac{7e}{n}\right)^n z^n$.

23. $f(z)=\cos\sqrt{z}$.

24. $f(z)=e^z\cos z$.

25. $f(z)=\sum_{n=1}^{\infty}e^{-n^2}z^n$.

26. $f(z)=\sum_{n=1}^{\infty}\left(\frac{1}{n\ln n}\right)^n z^n$.

27. $f(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^{n^2}}$.

28. $f(z)=\sum_{n=0}^{\infty}\frac{\operatorname{ch}\sqrt{n}}{n!}z^n$.

29. $f(z)=\int_0^1 e^{zt^2}dt$.

30. $f(z)=\sum_{n=1}^{\infty}\left(\frac{z}{n}\right)^n$.

2. Find the indicator function of an entire function f:

1. $f(z)=e^{-(1-5i)z}$.

2. $f(z)=e^{(1+2i)z}$.

3. $f(z)=e^{2z^3}$.

4. $f(z)=\sin z$.

5. $f(z)=\cos z$.

6. $f(z)=e^{2iz^3}$.

7. $f(z)=e^{iz^4}$.

8. $f(z)=e^{(1+i)z^3}$.

9. $f(z)=e^{-2iz^3}$.

10. $f(z)=e^{iz^4}$.

11. $f(z)=e^{(1+i)z^4}$.

12. $f(z)=e^{(1-i)z^5}$.

13. $f(z)=\operatorname{sh} z$.

14. $f(z)=\operatorname{ch} z$.

15. $f(z)=e^{z^n}$, $n\in\mathbb{N}$.

16. $f(z)=e^z+z^2$.

17. $f(z)=e^{az}$, $a=\alpha e^{i\varphi}$.

18. $f(z)=e^{(a-ib)z^\rho}$, $\rho>0$.

19. $f(z)=e^{2z+1}$.

20. $f(z)=e^{-2z+3}$.

21. $f(z)=\frac{\sin\sqrt{z}}{\sqrt{z}}$

22. $f(z)=\frac{1}{\Gamma(z)}$.

23. $f(z)=e^{-z}+z$.

24. $f(z)=e^{(a+ib)z^\rho}$, $\rho>0$.

25. $f(z)=ze^z$.

26. $f(z)=e^{-z^2}+z^2$.

27. $f(z)=e^z\cos z$.

28. $f(z)=e^{-z}\sin z$.

29. $f(z)=e^z+1+z$.

30. $f(z)=e^{z^2}+1+z^2$.

3. Find the Borel transform of the function L:

1. $L(z)=e^{-2iz}$.

2. $L(z)=e^{-iz}-e^z$.

3. $L(z) = \operatorname{ch} z$.

4. $L(z) = \operatorname{sh} z$.

5. $L(z) = e^{-iz} - e^{z} + e^{-z} + e^{iz}$.

6. $L(z) = e^{iz} + z$.

7. $L(z) = \cos\sqrt{z}$.

8. $L(z) = e^{-iz} - iz$.

9. $L(z) = \dfrac{\sin z}{z}$.

10. $L(z) = \dfrac{\sin\sqrt{z}}{\sqrt{z}}$.

11. $L(z) = z - 2z^2$.

12. $L(z) = z + e^{-2z}$.

13. $L(z) = \sin\sqrt{z}$.

14. $L(z) = Ae^{az}$, $A, a \neq 0$.

15. $L(z) = \cos z \cos(iz)$.

16. $L(z) = \sin(iz)$.

17. $L(z) = \cos(iz)$.

18. $L(z) = 1 + e^{z}$.

19. $L(z) = \operatorname{ch} z + \sin z$.

20. $L(z) = e^{z-1}$.

21. $L(z) = e^{z} + \sin z$.

22. $L(z) = e^{-z} + \cos z$.

23. $L(z) = 1 + 2z + 3z^2$.

24. $L(z) = z + \operatorname{sh} z$.

25. $L(z) = z^2 + \operatorname{ch} z$.

26. $L(z) = \sin z \sin(iz)$.

27. $L(z) = e^{-2iz} + iz + e^{2iz}$.

28. $L(z) = e^{z} + \operatorname{sh} z$.

29. $L(z) = e^{z} + \operatorname{ch} z$.

30. $L(z) = e^{2iz} + e^{iz} + z$.

Chapter 4. Infinite products of entire functions of finite order and related problems

4.1. The convergence exponent of the sequence. Let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence of complex numbers such that $0<|\lambda_k|\nearrow+\infty$ as $k\to\infty$. The greatest lower bound τ of τ_1's such that the series [24, 31, 45, 47, 54]

$$\sum_{k=1}^{\infty}|\lambda_k|^{-\tau_1} \tag{1}$$

converge is called the convergence exponent of the sequence (λ_k). If $\tau<+\infty$, then for every $\tau_1>\tau$ the series (1) converges, while for $\tau_1=\tau$ the series (1) may be convergent or it may be divergent.

Theorem 1 ([24, 31, 45, 47, 54]). *For every sequence* (λ_k), *the convergence exponent* τ *can be found by the formula:*

$$\tau=\varlimsup_{k\to\infty}\frac{\ln k}{\ln|\lambda_k|}. \tag{2}$$

Proof. Let the right-hand side of (2) be denoted by $\tau^{\bullet}$. We need to show that $\tau=\tau^{\bullet}$. Let $\tau<+\infty$. Then, for every $\tau_1>\tau$, the series (1) converges. Therefore,

$$\frac{n-\frac{n}{2}}{|\lambda_n|^{\tau_1}}\le\sum_{k=[n/2]}^{n}\frac{1}{|\lambda_k|^{\tau_1}}\to 0,\ n\to\infty.$$

This yields that for every $\tau_1>\tau$

$$\varlimsup_{n\to\infty}\frac{\ln n}{\ln|\lambda_n|}<\tau_1.$$

Hence, $\tau\le\tau^{\bullet}$. Now let $\tau^{\bullet}<+\infty$. Taking an arbitrary numbers τ_1 and τ_2 such that $\tau^{\bullet}<\tau_2<\tau_1<+\infty$, we have $1/|\lambda_k|<(1/k)^{\tau_2}$ for $k\ge k^{\bullet}$. From this, it follows that the series (1) converges for every $\tau_1>\tau^{\bullet}$. Therefore, $\tau^{\bullet}\le\tau$. Thus, $\tau^{\bullet}=\tau$. Theorem 1 is proved.►

Example 1. *If* $\lambda_k=k$, *then* $\tau=1$ *and for* $\tau_1=\tau$ *the series (1) diverges.*

Example 2. *If* $\lambda_k=(k+1)\ln^2(k+1)$, *then* $\tau=1$ *and for* $\tau_1=\tau$ *the series (1) converges.*

Example 3. *Let* $\rho\in(0;+\infty)$ *and* $\lambda_k=k^{1/\rho}$. *Then* $\tau=\rho$.

Let $n(t)$ be the counting function of a sequence (λ_k), i.e., $n(t)=\sum_{|\lambda_k|\le t}1=\max\{k:|\lambda_k|\le t\}$, and

$$N(r)=\int_0^r \frac{n(t)-n(0)}{t}dt+n(0)\ln r\,,\ r>0\,.$$

Theorem 2 ([24, 31, 45, 47, 54]). *For every sequence* (λ_k) *holds*

$$\overline{\lim_{k\to\infty}}\frac{\ln k}{\ln|\lambda_k|}=\overline{\lim_{r\to+\infty}}\frac{\ln n(r)}{\ln r}\,.$$

Proof. Indeed,

$$\frac{\ln k}{\ln|\lambda_k|}\le\frac{\ln n\left(|\lambda_k|\right)}{\ln|\lambda_k|}\,.$$

From the other hand, for every $r\ge|\lambda_1|$ there exists k such that $|\lambda_k|\le r<|\lambda_{k+1}|$ and

$$\frac{\ln n(r)}{\ln r}=\frac{\ln k}{\ln r}\le\frac{\ln k}{\ln|\lambda_k|}\,.$$

Theorem 2 is proved. ►

Theorem 3. *For every sequence* (λ_k) *holds*

$$\overline{\lim_{r\to+\infty}}\frac{\ln n(r)}{\ln r}=\overline{\lim_{r\to+\infty}}\frac{\ln N(r)}{\ln r}\,.$$

Proof. The required statement follows from the inequalities:

$$N(r)=\int_{|\lambda_1|}^{r}\frac{n(t)}{t}dt\le n(r)\ln\frac{r}{|\lambda_1|}\,,$$

$$N\left(r\right)\ge\int_{r/e}^{r}\frac{n(t)}{t}dt\ge n\left(\frac{r}{e}\right),\quad r\ge e|\lambda_1|\,.$$

Theorem 3 is proved. ►

Theorem 4 ([24, 31, 45, 47, 54]). *The series (1) and the integral*

$$\int_1^{+\infty}\frac{n(t)}{t^{\tau_1+1}}dt \tag{3}$$

converge and diverge simultaneously.

Proof. Since

$$\int_r^{+\infty}\frac{n(t)}{t^{\tau_1+1}}dt\ge\frac{n(r)}{\tau_1 r^{\tau_1}}\,,$$

from the convergence of the integral (3) it follows that $n(r)=o(r^{\tau_1})$ as $r\to+\infty$. Therefore, the required conclusion follows from the equality:

$$\sum_{|\lambda_1|\le r}\frac{1}{|\lambda_k|^{\tau_1}}=\int_{|\lambda_1|}^{r}\frac{dn(t)}{t^{\tau_1}}=\frac{n(r)}{r^{\tau_1}}-\frac{n(|\lambda_1|)}{|\lambda_1|^{\tau_1}}+\tau_1\int_{|\lambda_1|}^{r}\frac{n(t)dt}{t^{\tau_1+1}}.$$

Theorem 4 is proved. ►

4.2. The Weierstrass canonical product. Relation between the genus and the convergence exponent of the sequence of zeros. Let the sequence (λ_k) has a finite convergence exponent and let p be the smallest integer such that

$$\sum_{k=1}^{\infty}|\lambda_k|^{-p-1}<+\infty,$$

and

$$E(z/\lambda_k;p)=\left(1-\frac{z}{\lambda_k}\right)\exp\left(\sum_{j=1}^{p}\frac{z^j}{j\lambda_k^j}\right).$$

Then the product

$$L(z)=\prod_{k=1}^{\infty}E(z/\lambda_k;p)=\prod_{k=1}^{\infty}\left(1-\frac{z}{\lambda_k}\right)\exp\left(\sum_{j=1}^{p}\frac{z^j}{j\lambda_k^j}\right)$$

is called the Weierstrass canonical product of genus p. It is uniformly convergent on each compact set in $\mathbb{C}$, the function L is an entire and the sequence (λ_k) is a sequence of its zeros. In this case, the number p is called the genus of the sequence (λ_k) or the genus of a canonical product.

Theorem 1 ([24, 31, 45, 47, 54]). *For every sequence* (λ_k) *holds* $p\le\tau\le p+1$. *If* τ *is a noninteger, then* $p=[\tau]$.

Proof. This theorem follows directly from the definitions. ►

Example 1. *If* $\lambda_n=n$, *then* $p=\tau=1$ *and* $L(z)=\prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)e^{\frac{z}{n}}$ *is the corresponding Weierstrass canonical product.*

Example 2. *If* $\lambda_n=(n+1)\ln^2(n+1)$, *then* $\tau=1$, $p=0$ *and* $L(z)=\prod_{n=1}^{\infty}\left(1-\frac{z}{(n+1)\ln^2(n+1)}\right)$ *is the corresponding Weierstrass canonical product.*

Example 3. *If* $\lambda_n = n^{2/7}$, *then* $\tau = 7/2$, $p=3$ *and* $L(z)=\prod_{n=1}^{\infty}\left(1-z/n^{2/7}\right)\exp\left(\frac{z}{n^{2/7}}+\frac{z^2}{2n^{4/7}}+\frac{z^3}{3n^{6/7}}\right)$ *is the corresponding Weierstrass canonical product.*

4.3. Relation between the order of a canonical product and the convergence exponent of its zeros. Lower estimates for canonical products.

Theorem 1 ([24, 31, 45, 47]). *The order of the canonical product is equal to the convergence exponent of the sequence* (λ_k): $\rho_L = \tau$.

Proof. It follows from the Jensen's inequality $N(r) \le \ln M_L(r)$ that $\tau \le \rho_L$. To prove the opposite inequality, we choose the number τ_1 such that $\tau \le \tau_1 \le p+1$ and $\sum_{k=1}^{\infty}|\lambda_k|^{-\tau_1} < +\infty$. Then, taking into account that $p < \tau_1 \le p+1$, we obtain

$$\left|E\left(z/\lambda_k;p\right)\right| \le \exp\left(2|z/\lambda_k|^{p+1}\right) \le \exp\left(2|z/\lambda_k|^{\tau_1}\right),\ |z/\lambda_k| \le 1/2,$$

$$\left|E\left(z/\lambda_k;p\right)\right| \le \exp\left(\ln|1+z/\lambda_k| + \left(2|z|/|\lambda_k|\right)^p\right) \le$$

$$\le c_0\left(2|z|/|\lambda_k|\right)^p \le c_1|z/\lambda_k|^{\tau_1},\ |z/\lambda_k| \ge 1/2 .$$

Hence, $\rho_L \le \tau$ and the theorem is proved. ►

Corollary 1. *In order that the sequence* (λ_k) *be a sequence of zeros of some entire function* f *of order* $\rho_f \le \rho$, *it is necessary and sufficient that*

$$\overline{\lim_{r\to+\infty}}\frac{\ln N(r)}{\ln r} \le \rho .$$

Example 1. *If* $\lambda_n = \sqrt{n}$, *then* $\tau = 2$ *and the order of the Weierstrass canonical product* $L(z)=\prod_{n=1}^{\infty}\left(1-z/\sqrt{n}\right)\exp\left(\frac{z}{\sqrt{n}}+\frac{1}{2}\left(\frac{z}{\sqrt{n}}\right)^2\right)$ *is equal to* 2.

Theorem 2 ([24, 31, 45, 47]). *For the Weierstrass canonical product* L *of order* $\rho \ne +\infty$ *holds*

$$\left(\forall\varepsilon>0\right)\left(\exists r_0\right)\left(\forall z \notin U,\ |z| \ge r_0\right): \left|L(z)\right| \ge \exp\left(-|z|^{\rho+\varepsilon}\right),$$

for every $\rho_1 > \rho$, *where* $U = \bigcup_n \left\{z : |z-\lambda_n| \le |\lambda_n|^{-\rho_1}\right\}$.

Proof. Indeed, we have

$$\ln|L(z)| \geq \sum_{|\lambda_n|\leq 2|z|} \ln\left|1-\frac{z}{\lambda_n}\right| - \sum_{|\lambda_n|\leq 2|z|} \left\|\ln\left|\sum_{k=1}^{p} z^k \Big/ k\lambda_n^k\right\|\right| -$$

$$-\sum_{|\lambda_n|\leq 2|z|} \left|\ln|E(z/\lambda_n;p)|\right| \geq c_0|z|^{\tau_0}\sum_{n=1}^{\infty} 1\Big/|\lambda_n|^{\tau_0} + \sum_{|\lambda_n|\geq 2|z|} \ln|\lambda_n|^{-1-\rho_1} + O(1) \geq$$

$$\geq c_1|z|^{\tau_0} - n(2|z|)\ln(2|z|)^{1+\rho_1} + O(1),$$

which proves the theorem. ►

Corollary 2. *For the Weierstrass canonical product* L *of order* $\rho \neq +\infty$ *there exists a sequence* (r_k), $0 < r_k \uparrow +\infty$, *such that*

$$(\forall \varepsilon > 0)(\exists k^*)(\forall k \geq k^*)(\forall \varphi \in [0;2\pi]) : \left|L(r_k e^{i\varphi})\right| \geq \exp\left(-r_k^{\rho+\varepsilon}\right).$$

Proof. To prove this corollary, it is necessary to take into account that $\sum_{k=1}^{\infty} |\lambda_k|^{-\rho_1} < +\infty$. ►

4.4. Expansion of entire functions of finite order into infinite products. Genus of an entire function. One of the main theorems of the theory of entire functions is

Theorem 1 (Hadamard-Borel) ([24, 31, 45, 47, 54]). *Every entire function* $f \neq 0$ *of finite order* ρ *may be represented in the form*

$$f(z) = z^m e^{Q(z)} \prod_{k=1}^{\infty}\left(1-\frac{z}{\lambda_k}\right)\exp\left(\sum_{j=1}^{p}\frac{z^j}{j\lambda_k^j}\right), \tag{1}$$

where (λ_k) *is a sequence of zeros of the function* f, $m \in \mathbb{Z}_+$ *is the multiplicity of the zero at the origin,* $Q(z) = \sum_{i=0}^{\nu} Q_i z^i$ *is a polynomial of degree* $\nu \leq \rho$, $p \leq \rho$ *is the smallest integer for which*

$$\sum_{k=1}^{\infty} |\lambda_k|^{-p-1} < +\infty. \tag{2}$$

Conversely, if Q *a polynomial of degree* $\nu \leq \rho$, $m \in \mathbb{Z}_+$, (λ_k) *is a sequence with convergence exponent* $\tau \leq \rho$ *and* $p \leq \rho$ *is the smallest integer such that holds (2), then the function (1) is entire and has the order* $\rho = \max\{\nu;\tau\}$.

Proof. It follows from the Jensen's inequality that the convergence exponent τ of the sequence of zeros of f does not exceed ρ. By the Weierstrass theorem, the function f has the form (1). We prove that Q is a polynomial of degree $\nu \leq \rho$. Let us denote the last factor in (1) by $L(z)$. Then by Corollary 2 from the section 4.3 there exists a sequence (r_k), $0<r_k\uparrow+\infty$, such that

$$(\forall\varepsilon>0)\left(\exists k^\bullet\right)\left(\forall k\ge k^\bullet\right):\max_{|z|=r_k}\left|e^{Q(z)}\right|=\max_{|z|=r_k}\left|\frac{f(z)}{z^m L(z)}\right|\le\exp\left(r_k^{\rho+\varepsilon}\right)$$

and $\max\limits_{|z|=r_k}\operatorname{Re}Q(z)\le r_k^{\rho+\varepsilon}$. Thus, applying the Schwarz formula for $R=r_k$, we obtain $\left|Q^{(n)}(0)\right|\le c_1 r_k^{\rho+\varepsilon-n}$. It follows that $Q^{(n)}(0)=0$ for $n>\rho$, that is Q is a polynomial of degree $\nu\le\rho$. The second part of Theorem 1 is also valid because the inequality $\rho\le\max\{\nu;\tau\}$ is obvious, and the inequality $\rho\ge\max\{\nu;\tau\}$ was established above. Theorem 1 is proved. ►

Corollary 1. *The order ρ of an entire function f having the representation (1) is determined as follows:* $\rho=\max\{\nu;\tau\}$.

Corollary 2. *The order of an entire function of noninteger order is equal to the convergence exponent of the sequence of its zeros.*

Theorem 2 ([24, 31, 45, 47, 54]). *The following equalities are valid:*

$$\sin z=z\prod_{k\in\mathbb{Z}\setminus\{0\}}\left(1-\frac{z}{k\pi}\right)e^{\frac{z}{k\pi}}=z\prod_{k=1}^{\infty}\left(1-\frac{z^2}{(k\pi)^2}\right),\qquad (3)$$

$$\cos z=\prod_{k=-\infty}^{+\infty}\left(1-\frac{z}{(\pi/2+\pi k)}\right)e^{\frac{z}{\pi/2+\pi k}}=\prod_{k=0}^{\infty}\left(1-\frac{z^2}{(\pi/2+\pi k)^2}\right).\qquad (4)$$

Moreover, the last products converge uniformly on every compact set in $\mathbb{C}$.

Proof. Indeed, the function $z^{-1/2}\sin\sqrt{z}$ is an entire function of order $\rho=1/2$ with zeros at the points $k^2\pi^2$, $k\in\mathbb{N}$. Therefore, by the Hadamard-Borel theorem, we have

$$\frac{\sin\sqrt{z}}{\sqrt{z}}=c_0\prod_{k=1}^{\infty}\left(1-\frac{z}{k^2\pi^2}\right).$$

Taking $z=0$ we conclude that $c_0=0$. This yields (3). The function $\cos\sqrt{z}$ is an entire function of order $\rho=1/2$, which vanishes at the points $(\pi k+\pi/2)^2$, $k\in\mathbb{Z}_+$. According to the Hadamard-Borel theorem, we obtain

$$\cos\sqrt{z}=c_0\prod_{k=0}^{\infty}\left(1-\frac{z}{(\pi/2+\pi k)^2}\right).$$

Putting $z=0$ here, we get $c_0=0$. Thus, the equality (4) is true. ►

The genus of an entire function f of order $\rho \in [0;+\infty)$ is called [24, 31, 45, 47, 54] the number $q = \max\{p;\nu\}$, where p is the genus of the sequence (λ_k) and ν is the degree of the polynomial Q in the representation (1).

Theorem 3 (Poincaré) ([24, 31, 45, 47, 54]). *For every entire function f of order $\rho \in [0;+\infty)$ holds $q \le \rho \le q+1$. If ρ is a noninteger number, then $q = p = [\rho]$.*

Proof. Since $p \le \tau \le p+1$ and $\rho = \max\{\nu;\tau\}$, the statement of Theorem 3 is valid. ▶

Example 1. *If* $f(z) = z^4 e^{2z+z^3} \prod_{n=1}^{\infty} (1 - z/n) e^{z/n}$ *, then* $q = 3$.

Example 2. *If* $f(z) = z^4 e^{2z} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\sqrt{n}}\right) \exp\left(\frac{z}{\sqrt{n}} + \frac{1}{2}\left(\frac{z}{\sqrt{n}}\right)^2\right)$*, then* $q = 2$.

4.5. The Fourier coefficients of an entire function. The functions

$$C_k(R) = \frac{1}{2\pi} \int_0^{2\pi} \ln\left| f(Re^{i\theta})\right| e^{-ik\theta} d\theta, \ k \in \mathbb{Z}, \quad R > 0,$$

are called the Fourier coefficients of an entire function f.

Theorem 1 ([13, 23, 31, 45]). *If $f \ne 0$ is an entire function, then*

$$C_0(R) = N(R) + \ln|f_m|, \tag{1}$$

$$C_k(R) = \overline{C}_{-k}(R) = \frac{1}{2}\alpha_k R^k + \frac{1}{2k}\left(\sum_{0<|\lambda_n|\le R} \left(\frac{R}{\lambda_n}\right)^k - \left(\frac{\overline{\lambda}_n}{R}\right)^k \right), \ k \ge 1, \tag{2}$$

where (λ_n) is a sequence of zeros of the function f, m is the multiplicity of the zero at the origin and the numbers α_k are determined from the expansion

$$\ln \frac{f(z)}{f_m z^m} = \sum_{k=1}^{\infty} \alpha_k z^k ,$$

and $\ln w$ is a branch of the logarithm in $\mathbb{C}_+$ that takes the value zero at the point 1.

Proof. The equality (1) follows from Jensen's equality. Let us prove (2). Let

$$D = \mathbb{C} \setminus \bigcup_k \left\{ z : \arg z = \varphi_k, |z| \ge |\lambda_1| \right\},$$

where $\varphi_k = \arg\lambda_k \in [0;2\pi]$. The function $f(z)/(f_m z^m)$ has no zeros in D. Therefore, using the Poisson-Jensen formula, we conclude that the holomorphic in the disk $\{z:|z|<R\}$ function

$$F(z)=\frac{1}{2\pi}\int_0^{2\pi}\ln\left|f(Re^{i\theta})\right|\frac{Re^{i\theta}+z}{Re^{i\theta}-z}d\theta+\sum_{0<|\lambda_n|\le R}\ln\frac{R(z-\lambda_n)}{R^2-z\bar{\lambda}_n}-\ln\left|f_m\right|R^m$$

coincides with one of the branches of the function $\ln(f(z)/f_m z^m)$. Moreover,

$$F^{(k)}(z)=\frac{k!}{2\pi}\int_0^{2\pi}\ln\left|f(Re^{i\theta})\right|\frac{2Re^{i\theta}}{(Re^{i\theta}-z)^{k+1}}d\theta+$$

$$+(k-1)!\sum_{0<|\lambda_n|\le R}\left(\frac{\bar{\lambda}_n^k}{(R^2-z\bar{\lambda}_n)^k}-\frac{(-1)^k}{(z-\lambda_n)^k}\right),\quad k\ge 1,\quad |z|<R,$$

$$\frac{F^{(k)}(0)}{2k!}=\frac{1}{2\pi}\int_0^{2\pi}\ln\left|f(Re^{i\theta})\right|e^{-ik\theta}d\theta+\frac{1}{k}\sum_{0<|\lambda_n|\le R}\left(\left(\frac{R}{\lambda_n}\right)^k-\left(\frac{\bar{\lambda}_n}{r}\right)^k\right),\quad k\ge 1,$$

It follows that the formula (2) holds for $k\ge 1$. The equality $C_k(R)=\overline{C}_{-k}(R),\quad k\le 1$, is obtained directly from the definition $C_k(R)$. Theorem 1 is proved. ►

Example 1. *If* $f(z)=e^z$, *then* $C_1(R)=C_{-1}(R)=R/2$ *and* $C_k(R)=0$ *for* $k\in\mathbb{Z}\setminus\{-1;1\}$.

Example 2. *If* $\sum_{n=1}^{\infty}1/|\lambda_n|<+\infty$ *and* $f(z)=\prod_{n=1}^{\infty}(1-z/\lambda_n)$, *then we have:*

$$\ln f(z)=\sum_{n=1}^{\infty}\ln\left(1-\frac{z}{\lambda_n}\right)=-\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{z}{\lambda_n}\right)^k=-\sum_{k=1}^{\infty}\frac{z^k}{k}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^k},$$

$$\alpha_0=0,\ C_0(R)=N(R),$$

$$\alpha_k=-\frac{1}{k}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^k},\quad k\ge 1,$$

$$C_k(R)=-\frac{1}{2k}\sum_{|\lambda_n|>R}\left(\frac{R}{\lambda_n}\right)^k-\frac{1}{2k}\sum_{0<|\lambda_n|\le R}\left(\frac{\bar{\lambda}_n}{R}\right)^k,\quad k\ge 1,$$

$$C_k(R)=-\frac{1}{2k}\sum_{|\lambda_n|>R}\left(\frac{R}{\bar{\lambda}_n}\right)^k-\frac{1}{2k}\sum_{0<|\lambda_n|\le R}\left(\frac{\lambda_n}{R}\right)^k,\quad k\le -1.$$

Example 3. *If* $\sum_{n=1}^{\infty} 1/|\lambda_n|^2 < +\infty$ *and*

$$f(z) = e^{Q_1 z} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right) e^{\frac{z}{\lambda_n}} ,$$

then

$$\ln f(z) = Q_1 z + \sum_{n=1}^{\infty} \left(\ln\left(1 - \frac{z}{\lambda_n}\right) + \frac{z}{\lambda_n} \right) =$$

$$= Q_1 z - \sum_{n=1}^{\infty} \sum_{k=2}^{\infty} \frac{1}{k} \left(\frac{z}{\lambda_n}\right)^k = Q_1 z - \sum_{k=2}^{\infty} \frac{z^k}{k} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^k} ,\ \alpha_0 = 0 ,$$

$$\alpha_k = Q_k ,\ C_0(R) = N(R) ,$$

$$C_1(R) = \frac{1}{2} Q_1 R + \frac{1}{2} \sum_{0<|\lambda_n|\le R} \left(\frac{R}{\lambda_n} - \frac{\overline{\lambda}_n}{R} \right),$$

$$C_{-1}(R) = \frac{1}{2} \overline{Q_1} R + \frac{1}{2} \sum_{0<|\lambda_n|\le R} \left(\frac{R}{\overline{\lambda}_n} - \frac{\lambda_n}{R} \right),$$

$$\alpha_k = -\frac{1}{k} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^k} ,\quad k \ge 2 ,$$

$$C_k(R) = -\frac{1}{2k} \sum_{|\lambda_n|>R} \left(\frac{R}{\lambda_n}\right)^k - \frac{1}{2k} \sum_{0<|\lambda_n|\le R} \left(\frac{\overline{\lambda}_n}{R}\right)^k ,\quad k \ge 2 ,$$

$$C_k(R) = -\frac{1}{2k} \sum_{|\lambda_n|>R} \left(\frac{R}{\overline{\lambda}_n}\right)^k - \frac{1}{2k} \sum_{0<|\lambda_n|\le R} \left(\frac{\lambda_n}{R}\right)^k ,\quad k \le -2 .$$

Example 4. *Let* $Q(z) = \sum_{k=1}^{\infty} Q_k z^k$ *be an entire function and let* (p_n) *be a sequence of nonnegative integers such that the series* $\sum_{n=1}^{\infty} \left(r/|\lambda_n|\right)^{p_n+1}$, $\lambda_n \ne 0$, *converges for every* $r \in [0;+\infty)$. *Then*

$$f(z) = e^{Q(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right) \exp\left(\sum_{k=1}^{p_n} \frac{z^k}{k \lambda_n^k} \right)$$

is an entire function and we have:

$$\ln f(z)=\sum_{k=1}^{\infty}Q_k z^k+\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{-1}{k}\left(\frac{z}{\lambda_n}\right)^k+\sum_{k=1}^{p_n}\frac{1}{k}\left(\frac{z}{\lambda_n}\right)^k\right)=$$

$$=\sum_{k=1}^{\infty}Q_k z^k-\sum_{k=p_n+1}^{\infty}\frac{z^k}{k}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^k},\ \alpha_0=0,\ C_0(R)=N(R),$$

$$\alpha_k=Q_k,\ k\in\overline{1;p_n},$$

$$C_k(R)=\frac{1}{2}Q_k R^k+\frac{1}{2k}\sum_{0<|\lambda_n|\le R}\left(\left(\frac{R}{\lambda_n}\right)^k-\left(\frac{\bar{\lambda}_n}{R}\right)^k\right),\ k\in\overline{1;p_n},$$

$$C_k(R)=\frac{1}{2}\bar{Q}_k R^k+\frac{1}{2k}\sum_{0<|\lambda_n|\le R}\left(\left(\frac{R}{\bar{\lambda}_n}\right)^k-\left(\frac{\lambda_n}{R}\right)^k\right),\ k\in\overline{-p_n;-1},$$

$$\alpha_k=Q_k-\frac{1}{k}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^k},\ k>p_n,$$

$$C_k(R)=\frac{1}{2}Q_k R^k-\frac{1}{2k}\sum_{|\lambda_n|>R}\left(\frac{R}{\lambda_n}\right)^k-\frac{1}{2k}\sum_{0<|\lambda_n|\le R}\left(\frac{\bar{\lambda}_n}{R}\right)^k,\ k>p_n,$$

$$C_k(R)=\frac{1}{2}\bar{Q}_k R^k-\frac{1}{2k}\sum_{|\lambda_n|>R}\left(\frac{R}{\bar{\lambda}_n}\right)^k-\frac{1}{2k}\sum_{0<|\lambda_n|\le R}\left(\frac{\lambda_n}{R}\right)^k,\ k<-p_n.$$

4.6. Relation between the type of an entire function and its zeros. According to the Hadamard-Borel theorem, an entire function f of order $\rho\in(0;+\infty)$ may be represented in the form

$$f(z)=z^m e^{Q(z)}\prod_{k=1}^{\infty}E\left(\frac{z}{\lambda_k};p\right) \qquad (1)$$

where $Q(z)=\sum_{i=0}^{\nu}Q_i z^i$ is a polynomial of degree $\nu\le\rho$, $p\le\rho$ is the genus of the sequence (λ_k) of its zeros and $m\in\mathbb{Z}_+$ is the multiplicity of the zero of f at the origin and $E(w;p)$ is the Weierstrass primary factor. Let us determine when a function of order $\rho\in(0;+\infty)$ has a finite type.

Theorem 1 ([24, 31, 45, 47]). *In order that an entire function f of noninteger order $\rho\in(0;+\infty)$ has finite type, it is necessary and sufficient that the sequence (λ_k) has finite upper density:*

$$\tau_0 := \overline{\lim_{k\to\infty}} \frac{k}{|\lambda_k|^{\rho}} < +\infty\,. \qquad (2)$$

Proof. The necessity follows from Jensen's inequality. To prove the sufficiency, let us denote the canonical product from (1) by $L(z)$. Then

$$\ln|L(z)| \le \sum_{|\lambda_k|\le 2|z|} \ln\left(1+|z|/|\lambda_k|\right) + \sum_{|\lambda_k|\le 2|z|} 2|z|^p/|\lambda_k|^p + \sum_{|\lambda_k|>2|z|} 2^p|z/\lambda_k|^{p+1} \le$$

$$\le \sum_{|\lambda_k|\le 2|z|} \ln\frac{2|z|}{|\lambda_k|} + \sum_{|\lambda_k|\le 2|z|} \ln\left(\frac{1}{2}+\frac{|\lambda_k|}{2|z|}\right) + \sum_{|\lambda_k|\le 2|z|} 2\frac{|z|^p}{|\lambda_k|^p} + \sum_{|\lambda_k|>2|z|} 2^p\left|\frac{z}{\lambda_k}\right|^{p+1} +$$

$$+2|z|^p \int_{\lambda_1}^{2|z|} \frac{dn(t)}{t^p} + 2^p|z|^{p+1}\int_{2|z|}^{+\infty}\frac{dn(t)}{t^{p+1}} = N(2r) + n\left(2|z|\right)\ln\frac{3}{2} +$$

$$+2|z|^p \int_{\lambda_1}^{2|z|} \frac{dn(t)}{t^p} + 2^{\rho}|z|^{p+1}\int_{2|z|}^{+\infty}\frac{dn(t)}{t^{p+1}} = O(r^{\rho}),\ r\to+\infty\,.$$

It remains to remark that the first two factors in (1) have order less than ρ. The theorem is proved. ►

Corollary 1. *The type of an entire function of noninteger order is equal to the type of the canonical product.*

Theorem 2 (Lindelöf) ([24, 31, 45, 47]). *In order that an entire function f of integer order $\rho\in(0;+\infty)$ is a function of finite type, it is necessary and sufficient that the condition (2) holds and*

$$\delta = \overline{\lim_{r\to+\infty}} \left| Q_{\rho} + \frac{1}{\rho}\sum_{0<|\lambda_k|\le r} \frac{1}{\lambda_k^{\rho}} \right| < +\infty\,. \qquad (3)$$

In this case, for $p=\rho$ the type of the function f is equal to zero if and only if $\delta=\tau_0=0$. If $p=\rho-1$, then the type of the function f is equal to the coefficient Q_{ρ} of z^{ρ} of the polynomial Q in the representation (1).

Proof. First, we will prove the necessity. The necessity of condition (2) follows from Jensen's inequality. To prove the necessity of condition (3), we will find the Fourier coefficients of the function f, assuming for simplicity that $f(0)=1$. Then

$$\ln f(z) = \sum_{k=1}^{\rho} Q_k z^k + \sum_{n=1}^{\infty}\ln\left(1-\frac{z}{\lambda_n}\right) + \sum_{k=1}^{p}\frac{z^k}{k\lambda_n^k} = \sum_{k=1}^{\rho} Q_k z^k - \sum_{n=1}^{\infty}\sum_{k=\rho+1}^{\infty}\frac{z^k}{k\lambda_n^k}\,.$$

Threfore

$$\alpha_k = \begin{cases} Q_k, & k \in \overline{1;p}, \\ -\sum_{n=1}^{\infty} \frac{1}{k\lambda_n^k}, & k \geq p+1. \end{cases}$$

Heence, $\alpha_\rho = Q_\rho$. Since condition (3) holds for $p = \rho - 1$, it is sufficient to consider the case $\rho = p$. Then

$$C_\rho(r) = \frac{1}{2} Q_\rho r^\rho + \frac{1}{2\rho} \sum_{|\lambda_n| \leq r} \left(\left(\frac{r}{\lambda_n} \right)^\rho - \left(\frac{\overline{\lambda}_n}{r} \right)^\rho \right).$$

Thus,

$$\delta(r) = r^{-\rho} \left(\frac{1}{2\pi} \int_0^{2\pi} \ln \left| f(re^{i\theta}) \right| e^{-i\rho\theta} d\theta + \frac{1}{2\rho} \sum_{|\lambda_n| \leq r} \left(\frac{\overline{\lambda}_n}{r} \right)^\rho \right), \qquad (4)$$

where

$$\delta(r) = \frac{1}{2} \left(Q_\rho + \frac{1}{\rho} \sum_{|\lambda_n| \leq r} \frac{1}{\lambda_n^\rho} \right).$$

It follows from the Jensen inequality that

$$N(r) + \frac{1}{2\pi} \int_0^{2\pi} \ln^+ \frac{1}{\left| f(re^{i\theta}) \right|} d\theta = \frac{1}{2\pi} \int_0^{2\pi} \ln^+ \left| f(re^{i\theta}) \right| d\theta \leq \ln M_f(r).$$

Further, taking into account that $\left| \ln |f| \right| = \ln^+ |f| + \ln^+ \frac{1}{|f|}$, we obtain

$$\left| \frac{1}{2\pi} \int_0^{2\pi} \ln \left| f(re^{i\theta}) \right| e^{-i\rho\theta} d\theta \right| \leq 2 \ln M_f(r). \qquad (5)$$

Furthermore,

$$\sum_{|\lambda_n| \leq r} \left(\frac{|\lambda_n|}{r} \right)^\rho = \frac{1}{r^\rho} \int_0^r t^\rho dn(t) = n(r) - \frac{\rho}{r^\rho} \int_0^r t^{\rho-1} n(t) dt.$$

This, together with (4) and (5), completes the proof of the necessary part of the theorem. Let us prove sufficiency. First, suppose that $\rho = p$ and $r = |z|$. Then

$$L(z):=\prod_{|\lambda_n|>0} E\left(\frac{z}{\lambda_n},\rho\right)=\exp\left(\frac{z^\rho}{\rho}\sum_{|\lambda_n|\le 2r}\frac{1}{\lambda_n^\rho}\right)\prod_{|\lambda_n|\le 2r} E\left(\frac{z}{\lambda_n},\rho-1\right)\times\prod_{|\lambda_n|>2r} E\left(\frac{z}{\lambda_n},\rho\right),$$

$$\left|\sum_{0<|\lambda_n|\le 2r}\frac{1}{\lambda_n^\rho}\right|\le\frac{n(2r)}{(2r)^\rho}+\rho\int_0^{2r}\frac{n(t)}{t^{\rho+1}}dt,$$

$$\sum_{|\lambda_n|\le 2r}\ln\left|E\left(\frac{z}{\lambda_n},\rho-1\right)\right|+\sum_{|\lambda_n|>2r}\ln\left|E\left(\frac{z}{\lambda_n},\rho\right)\right|\le$$

$$\le 2\sum_{|\lambda_n|\le 2r}\frac{r^{\rho-1}}{|\lambda_n|^{\rho-1}}+N(2r)+n(2r)\ln\frac{3}{2}+\sum_{|\lambda_n|>2r}\frac{(2r)^{\rho+1}}{|\lambda_n|^{\rho+1}}.$$

This implies a sufficient part of the theorem for $\rho=p$. Let $p=\rho-1$. Then

$$L(z):=\prod_{|\lambda_n|>0} E\left(\frac{z}{\lambda_n},\rho-1\right)=$$

$$=\exp\left(\frac{-z^\rho}{\rho}\sum_{|\lambda_n|\ge 2r}1/\lambda_n^\rho\right)\prod_{|\lambda_n|\le 2r} E\left(\frac{z}{\lambda_n},\rho-1\right)\prod_{|\lambda_n|>2r} E\left(\frac{z}{\lambda_n},\rho\right).$$

In this case, by using the relation $n(t)=o(t^\rho)$ as $t\to+\infty$ and the inequality

$$\left|\sum_{|\lambda_n|>2r}\frac{1}{\lambda_n^\rho}\right|\le\frac{n(2r)}{(2r)^\rho}+\rho\int_{2r}^{+\infty}\frac{n(t)}{t^{\rho+1}}dt,$$

we arrive to the completion of the proof of the theorem. ►

Example 1. *The function* $f(z)=\prod_{k=1}^{\infty}(1-z/k)e^{z/k}$ *is an entire function for which* $\lambda_k=k$, $\rho=\tau=1$ *and* $\sum_{0<|\lambda_k|\le R}1/\lambda_k\to+\infty$ *as* $R\to+\infty$. *Therefore* $\sigma=+\infty$.

Приклад 2. *Let*

$$\lambda_k=\begin{cases}-n, & k=2n-1,\\ n, & k=2n.\end{cases}$$

Then

$$f(z)=\prod_{n=-\infty,\,n\ne 0}^{\infty}(1-z/n)e^{z/n}=\prod_{k=1}^{\infty}(1-z/\lambda_k)e^{z/\lambda_k}$$

is an entire function such that $\rho=\tau=1$, $p=1$, $\tau_0=2$ *and* $\sum\limits_{0<|\lambda_k|\le R} 1/\lambda_k=0$.

Hence, f *is an entire function of finite type* σ. *Since* $f(z)=\dfrac{\sin\pi z}{\pi z}$, *we have* $\sigma=\pi$.

Example 3. *The function*

$$f(z)=\prod_{k=1}^{\infty}\left(1-\frac{z}{\sqrt{k}}\right)\exp\left(\frac{z}{\sqrt{k}}+\frac{1}{2}\left(\frac{z}{\sqrt{k}}\right)^2\right)$$

is an entire function for which $\lambda_k=\sqrt{k}$, $\rho=\tau=2$, $\tau_0=1$ *and* $\sum\limits_{0<|\lambda_k|\le R} 1/\lambda_k^{\rho}\to+\infty$ *as* $R\to+\infty$. *Thus,* $\sigma=+\infty$.

Example 4. *If* $\lambda_{2n-1}=\sqrt{n}$ *and* $\lambda_{2n}=i\sqrt{n}$, *then*

$$f(z)=\prod_{k=1}^{\infty}\left(1-\frac{z}{\lambda_k}\right)\exp\left(\frac{z}{\lambda_k}+\frac{1}{2}\left(\frac{z}{\lambda_k}\right)^2\right)$$

is an entire function such that $\rho=\tau=2$, $p=2$, $\tau_0=2$ *and* $\sum\limits_{0<|\lambda_n|\le R} 1/\lambda_n^{\rho}=0$. *Hence,* $\sigma\in(0;+\infty)$.

Example 5. *Let* $\lambda_{3k-2}=\sqrt{k}$, $\lambda_{3k-1}=\sqrt{k}e^{i2\pi/3}$ *and* $\lambda_{3k}=\sqrt{k}e^{i4\pi/3}$. *Then*

$$f(z)=\prod_{n=1}^{\infty}\left(1-\frac{z}{\lambda_n}\right)\exp\left(\frac{z}{\lambda_n}+\frac{1}{2}\left(\frac{z}{\lambda_n}\right)^2+\frac{1}{3}\left(\frac{z}{\lambda_n}\right)^3\right)$$

is an entire function for which $\rho=\tau=2$, $p=3$, $\tau_0=3$ *and* $\sum\limits_{0<|\lambda_n|\le R} 1/\lambda_n^{\rho}=0$, *because* $1+e^{-i4\pi/3}+e^{-i8\pi/3}=0$. *Therefore,* $\sigma\in(0;+\infty)$.

Example 6. *The function*

$$f(z)=\prod_{k=1}^{\infty}\left(1-z/(k\ln^2(2+k))\right)$$

is an entire function such that $\lambda_k=k\ln^2(2+k)$, $\rho=\tau=1$, $p=\rho-1=0$, $\tau_0=0$ *and* $Q_\rho=0$. *Thus,* $\sigma=0$.

Example 7. *The function*

$$f(z)=e^{3z^2}\prod_{k=1}^{\infty}\left(1-z/(k\ln^2(2+k))\right)$$

is an entire function for which $\lambda_k=k\ln^2(2+k)$, $\tau=1$, $\nu=2$, $\rho=\max\{1;2\}=2$, $p=\rho-1=0$, $\tau_0=0$ *and* $Q_\rho=3$. *Hence,* $\sigma=3$.

4.7. Asymptotic properties of entire functions of finite order.

Theorem 1 ([13, 30, 31, 45-47]). *Let* $\Delta\in[0;+\infty)$, *let* $\rho\in(0;+\infty)$ *be a noninteger number,* $p=[\rho]$ *and let* (λ_n) *be a sequence of positive numbers such that*

$$n(t)=\Delta t^\rho+o(t^\rho),\ t\to+\infty. \qquad (1)$$

Then

$$L(z)=\prod_{n=1}^{\infty}E\left(\frac{z}{\lambda_n};p\right),\ z=re^{i\varphi}, \qquad (2)$$

is an entire function and for every $\varphi\in(0;2\pi)$ *holds*

$$\ln\left|L(re^{i\varphi})\right|=\frac{\pi\Delta r^\rho}{\sin\pi\rho}\cos\rho(\varphi-\pi)+o(r^\rho),\ r\to+\infty.$$

Moreover, for each $\delta\in(0;\pi)$, *the relation (2) holds uniformly with respect to* $\varphi\in[\delta;2\pi-\delta]$, *and if the sign "* $=$ *" is replaced with "* $\le$ *", the corresponding analogue of (2) will hold uniformly with respect to* $\varphi\in[0;2\pi]$.

Proof. Indeed, we have

$$\ln|L(z)|=\mathrm{Re}\int_0^{+\infty}\ln E\left(\frac{z}{t};p\right)dn(t)=-\mathrm{Re}\left(z^{p+1}\int_0^{+\infty}n(t)\frac{dt}{t^{p+1}(t-z)}\right)=$$

$$=-\mathrm{Re}\left(z^{p+1}\int_0^{\infty}\frac{dt}{t^{p+1-\rho}(t-z)}\right)+o(r^\rho)=\mathrm{Re}\left(\frac{\pi\Delta}{\sin\pi\rho}e^{i\rho(\varphi-\pi)}r^\rho\right)+o(r^\rho)=$$

$$=\frac{\pi\Delta}{\sin\pi\rho}r^\rho\cos\rho(\varphi-\pi)+o(r^\rho),\ r\to+\infty,$$

because

$$\int_0^{+\infty}\frac{t^{\alpha-1}}{t-z}dt=-\pi\frac{e^{-i\pi\alpha}}{\sin\pi\alpha}z^{\alpha-1},\ z\notin[0;+\infty),\ 0<\alpha<1. \ ►$$

Theorem 2 ([13, 30, 31, 45-47]). *Let* $\rho\in\mathbb{N}$ *and* (λ_n) *be a sequence of complex numbers such that* $n(t)=O(t^\rho)$ *as* $t\to+\infty$. *Then for the product (2) holds*

$$\ln|L(z)| = \begin{cases} \mathrm{Re}\left(\dfrac{z^\rho}{\rho}\sum\limits_{|\lambda_n|\le r}\lambda_n^{-\rho}\right) + O(r^\rho), & \rho = p, \\ o(r^\rho), & \rho = p-1, \end{cases} \qquad (3)$$

as $r \to +\infty$ *for every* $\varphi \in (0; 2\pi)$.

Proof. Indeed, the latter statement was established in the course of proving Lindelöf's theorem, and the rest follows from the estimates obtained there, taking into account that in the case $\rho = p$, as $r \to +\infty$:

$$\ln|L(z)| = \mathrm{Re}\left(\frac{z^\rho}{\rho}\sum_{|\lambda_n|\le 2r}\lambda_n^{-\rho}\right) + \sum_{|\lambda_n|\le 2r}\ln\left|1-\frac{z}{\lambda_n}\right| +$$

$$+\sum_{|\lambda_n|\le 2r}\ln\left|\sum_{k=1}^{\rho-1}\frac{z^k}{k\lambda_m^k}\right| + \sum_{|\lambda_n|>2r}\ln\left|E\left(\frac{z}{\lambda_n};\rho\right)\right| + \mathrm{Re}\left(\frac{z^\rho}{\rho}\sum_{|\lambda_n|\le 2r}\lambda_n^{-\rho}\right) + O(r^\rho).$$

In the case $\rho = p-1$ holds $n(t) = o(t^\rho)$ as $t \to +\infty$, and we have

$$\ln|L(z)| = \mathrm{Re}\left(-\frac{z^\rho}{\rho}\sum_{|\lambda_n|\le 2r}\lambda_n^{-\rho}\right) + \sum_{|\lambda_n|\le 2r}\ln\left|1-\frac{z}{\lambda_n}\right| +$$

$$+\sum_{|\lambda_n|\le 2r}\ln\left|\sum_{k=1}^{\rho-1}\frac{z^k}{k\lambda_n^k}\right| + \sum_{\lambda_n>2r}\ln\left|E\left(\frac{z}{\lambda_n};p\right)\right| = o(r^\rho),\ r \to +\infty. \ ►$$

Theorem 3 ([13, 30, 31, 45-47]). *Let* (λ_n) *be a sequence of positive numbers satisfying (1) for some* $\Delta \in [0; +\infty)$ *and* $\rho \in (0; +\infty)$. *Then for the function (2) holds*

$$\ln|L(z)| = \Lambda(z) + o(r^\rho),\ r \to +\infty, \qquad (4)$$

for every $\varphi \in (0; 2\pi)$ *and* $\rho \in \mathbb{N}$, *where*

$$\Lambda(z) = \begin{cases} \mathrm{Re}\left(\dfrac{z^\rho}{\rho}\sum\limits_{|\lambda_n|\le r}\lambda_n^{-\rho}\right) - \dfrac{\Delta}{\rho}r^\rho\cos\rho\varphi + (\pi-\varphi)\Delta r^\rho \sin\rho\varphi, & \rho = p, \\ 0, & \rho = p-1, \end{cases}$$

In this case, for every $\delta \in (0; \pi)$ *the relation (4) holds uniformly in* $\varphi \in [\delta; 2\pi - \delta]$.

Proof. Let

$$L_r(z) = \prod_{\lambda_n \le r} E\left(\frac{z}{\lambda_n};\rho-1\right) \times \prod_{\lambda_n > r} E\left(\frac{z}{\lambda_n};\rho\right).$$

Then

$$\ln|L_r(z)| = \mathrm{Re}\int_0^r \ln E\left(\frac{z}{t};\rho-1\right)dn(t) + \mathrm{Re}\int_r^{+\infty} \ln E\left(\frac{z}{t};\rho\right)dn(t) =$$

$$= -\frac{n(r)}{\rho}\cos\rho\varphi + r^{\rho}\left(\int_0^r \frac{\left(r\cos(\rho-1)\varphi - t\cos\rho\varphi\right)n(t)dt}{t^{\rho}(t^2-2tr\cos\varphi+r^2)} + \int_r^{+\infty} \frac{\left(r\cos\rho\varphi - t\cos(\rho+1)\varphi\right)n(t)dt}{t^{\rho+1}(t^2-2tr\cos\varphi+r^2)}\right) =$$

$$= -\frac{\tau r^{\rho}}{\rho}\cos\rho\varphi + \tau r^{\rho}\left(\int_0^1 \frac{\left(\cos(\rho-1)\varphi - u\cos\rho\varphi\right)du}{u^2-2u\cos\varphi+1} + \int_1^{+\infty} \frac{\left(\cos\rho\varphi - u\cos(\rho+1)\varphi\right)du}{u(t^2-2u\cos\varphi+1)}\right) =$$

$$= -\frac{\Delta}{\rho}r^{\rho}\cos\rho\varphi + (\pi-\varphi)\Delta r^{\rho}\sin\rho\varphi + o(r^{\rho}),\ r\to+\infty.$$

For $p=\rho$, we have

$$\ln\left|L(z)\right| = \mathrm{Re}\left(\frac{z^{\rho}}{\rho}\sum_{|\lambda_n|\le r}\lambda_n^{-\rho}\right) + \ln\left|L_r(z)\right|,$$

and from the previous equalities, we obtain (4). In the case $p=\rho+1$, the required statement is established in Theorem 2. ►

Remark 1. *If* $\lambda_n>0$ *and relation (1) holds, then*

$$\sum_{0<\lambda_n\le r}\frac{1}{\lambda_n^{\rho}} = \int_0^r \frac{dn(t)}{t^{\rho}} = \Delta\rho\ln r + o(\ln r),\ r\to+\infty.$$

For entire functions of order less than one, the mentioned relations are established quite easily. Let us present one simple fact.

Theorem 4 ([1]). *Let a sequence* (λ_k) *of complex numbers satisfy the condition*

$$(\exists\rho<1)(\forall\alpha>1)(\exists t_0)(\forall t_1\ge t_0)(t_2\ge t_1): \frac{n(t_2)}{t_2^{\rho}} \le \alpha\frac{n(t_1)}{t_1^{\rho}}, \qquad (5)$$

let $L(z)=\prod_{k=1}^{\infty}(1-z/\lambda_k)$ *and* $R=(\pi\rho/\sin(\pi\rho))^{1/\rho}$. *Then*

$$\ln M_L(r) \le N((1+o(1))Rr),\ r\to+\infty.$$

Proof. Let $\gamma>R$. Then

$$\ln M_L(r) \le \sum_{n=1}^{\infty}\ln\left(1+\frac{r}{|\lambda_n|}\right) = N(\gamma r) - \int_0^{\gamma r}\frac{n(t)dt}{t+r} + \int_{\gamma r}^{\infty}\frac{rn(t)dt}{t(t+r)}.$$

Using (5), we obtain

$$\int_{\gamma r}^{\infty}\frac{rn(t)}{t+r}dt\le\alpha\frac{n(\gamma r)}{(\gamma r)^{\rho}}\int_{\gamma r}^{+\infty}\frac{rt^{\rho-1}}{t+r}dt+c_q\,,\ r\ge r_0\,,$$

$$\int_{0}^{\gamma r}\frac{n(t)}{t+r}dt\ge\frac{1}{2}\frac{n(\gamma r)}{(\gamma r)^{\rho}}\int_{0}^{\gamma r}\frac{t^{\rho}}{t+r}dt+c_2\,,\ r\ge r_0\,.$$

Therefore, taking into account that

$$\int_{0}^{\infty}\frac{t^{\rho-1}}{1+t}dt=\frac{\pi}{\sin(\pi\rho)}\,,$$

we get

$$\ln M_L(r)\le N\left(\gamma r\right)+\alpha\frac{n(\gamma r)}{\gamma^{\rho}}\left(\int_{0}^{+\infty}\frac{t^{\rho-1}dt}{1+t}-\int_{0}^{\gamma}\frac{t^{\rho-1}(1+t)-t^{\rho}}{1+t}dt-\frac{1}{\alpha^2}\int_{0}^{\gamma}\frac{t^{\rho}dt}{1+t}\right)+c_4=$$

$$=N\left(\gamma r\right)+c_4+\frac{\alpha n(\gamma r)}{\rho\gamma^{\rho}}\left(R^{\rho}-\gamma^{\rho}+\rho\left(1-\frac{1}{\alpha^2}\right)\int_{0}^{\gamma}\frac{t^{\rho}dt}{1+t}\right). \qquad (6)$$

For a given $\gamma>R$, the number $\alpha>1$ can be chosen sufficiently close to 1 so that the expression in parentheses in formula (6) becomes negative. Thus, we arrive at the required conclusion. ►

Example 1. *Condition (5) is equivalent to the condition*

$$(\exists\rho<1)(\forall\alpha>1)(\exists n_0)(\forall k\ge n_0)(\forall n\ge k):|\lambda_k/\lambda_n|\le(\alpha k/n)^{1/\rho}\,. \qquad (7)$$

Indeed, if (7) holds and $t_2\ge t_1\ge\left|\lambda_{k_0}\right|$*, then for some* k *and* $m\ge k$*, we have* $\left|\lambda_m\right|\le t_2<\left|\lambda_{m+1}\right|$ *and* $\left|\lambda_k\right|\le t_2<\left|\lambda_{k+1}\right|$*. Therefore,*

$$\frac{n\left(t_2\right)}{t_2^{\rho}}\le\frac{m}{\left|\lambda_m\right|^{\rho}}\le\alpha\frac{k+1}{\left|\lambda_{k+1}\right|^{\rho}}\le\left(1+\frac{1}{k}\right)\alpha\frac{n\left(t_1\right)}{t_1^{\rho}}\,.$$

From this, the condition (5) is true. Conversely, let (5) holds and the numbers m *and* k *are such that* $m\ge k$ *and* $\left|\lambda_m\right|\ge\left|\lambda_k\right|\ge\alpha t_0$*. Then, for some* t_0 *and* t_1 *such that* $t_2>t_1>\alpha t_0$*, we obtaon* $\left|\lambda_m\right|\le t_2<\left|\lambda_{m+1}\right|$, $\left|\lambda_k\right|/\alpha<t_1<\left|\lambda_k\right|$ *and*

$$\frac{m}{\left|\lambda_m\right|^{\rho}}\le\alpha^{\rho}\frac{n\left(t_2\right)}{\left(\alpha\left|\lambda_m\right|\right)^{\rho}}\le\alpha^{\rho}\frac{n\left(t_2\right)}{t_2^{\rho}}\le\alpha^{\rho+1}\frac{n\left(t_1\right)}{t_1^{\rho}}\le\alpha^{2(1+\rho)}k/\left|\lambda_k\right|^{\rho}\,.$$

This implies (7).

Example 2. *Condition (7) holds if there exists a limit* $\lim_{n\to\infty}n/\left|\lambda_n\right|^{\rho}=\Delta\in(0;+\infty)$ *or if* $\left(\exists\Delta<1\right)\left(\forall n\right):\left|\lambda_n/\lambda_{n+1}\right|\le\Delta$.

In fact, since

$$(\forall\varepsilon>0)(\exists n_0)(\forall n\ge n_0):\ |1/\lambda_n|\le\left(1/(\tau-\varepsilon)n\right)^{1/\rho},$$

$$(\forall\varepsilon>0)(\exists k_0)(\forall k\ge k_0):\ |\lambda_k|\le\left((\tau+\varepsilon)k\right)^{1/\rho},$$

$$|\lambda_k/\lambda_n|=|\lambda_k/\lambda_{k+1}|\cdot|\lambda_{k+1}/\lambda_{k+2}|\cdot\ldots\cdot|\lambda_{n-1}/\lambda_n|\le$$

$$\le\left(k/(k+1)\right)^{1/\rho}\cdot\left((k+1)/(k+2)\right)^{1/\rho}\cdot\ldots\cdot\left((n-1)/n)\right)^{1/\rho}=\left(k/n\right)^{1/\rho},$$

for $n\ge k\ge n_0$.

Corollary 1 ([1]). *If*

$$(\forall\rho>0)(\forall\alpha>1)(\exists k_0)(\forall k\ge k_0)(\forall n\ge k):|\lambda_k/\lambda_n|\le(\alpha k/n)^{1/\rho},\qquad (8)$$

then

$$\ln M_L(r)=N((1+o(1))r),\ r\to+\infty.\qquad (9)$$

Proof. Indeed, the sufficiency follows from Jensen's inequality and Theorem 4, because $R\to 1$ when $\rho\to 0$. Now, let us prove the necessity. Suppose that $\arg\lambda_n=\text{const}$, condition (9) holds, but (8) does not hold. Then,

$$(\exists\rho>0)(\exists\alpha>1)(\forall n)(\exists k_n\ge n)(\exists m_n\ge k_n):|\lambda_{k_n}/\lambda_{m_n}|>(\alpha k_n/m_n)^{1/\rho}.$$

Since $|\lambda_{k_n}/\lambda_{m_n}|\le 1$, we have $m_n>\alpha k_n$. Let

$$p_n=\min\left\{s>k_n:|\lambda_{k_n}/\lambda_s|>(\alpha k_n/s)^{1/\rho}\right\},$$

where $s_n=[\mu p_n]$, $[x]$ is the integer part of a number $x>0$ and $1/\alpha<\eta<1$. Then $p_n>\alpha k_n$, $s_n>k_n$, and from the definition of p_n it follows that

$$|\lambda_{k_n}/\lambda_{p_n}|>(\alpha k_n/p_n)^{1/\rho},|\lambda_{k_n}/\lambda_{s_n}|\le(\alpha k_n/s_n)^{1/\rho}.$$

Therefore, the inequality $|\lambda_{s_n}/\lambda_{p_n}|\ge(s_n/p_n)^{1/\rho}>\eta_1$ holds for $0<\eta_1<\eta^{\rho}$. Let $i_n=\max\left\{k:|\lambda_k|\le 2|\lambda_{s_n}\right\}$ and $j_n=\max\{i_n;p_n\}$. In this case, we have $|\lambda_{s_n}/\lambda_{j_n}|\ge\eta_2$, where $\eta_2=\min\{1/2;\eta\}$. Hence,

$$\ln M_L\left(|\lambda_{s_n}|\right)=N\left(|\lambda_{s_n}|\right)+\sum_{\nu=1}^{s_n}\ln\left(1+|\lambda_\nu/\lambda_{s_n}|\right)+\sum_{\nu=s_n+1}^{\infty}\ln\left(1+|\lambda_{s_n}/\lambda_\nu|\right)\ge$$

$$\ge N\left(|\lambda_{s_n}|\right)+\sum_{\nu=s_n+1}^{j_n}\ln\left(1+|\lambda_{s_n}/\lambda_\nu|\right)\ge$$

$$\ge N\left(|\lambda_{s_n}|\right)+(j_n-s_n)\ln\left(1+|\lambda_{s_n}/\lambda_{j_n}|\right)\ge N\left(|\lambda_{s_n}|\right)+j_n\eta_3,\ \eta_3>0.$$

But $j_n\ge i_n=n(2|\lambda_{s_n}|)$. Thus, taking $\eta_4,\ 1<\eta_4<2,$ sufficiently close to 1 so that $\eta_3>\ln\eta_4$, we obtain

$$\ln M_L\left(\left|\lambda_{s_n}\right|\right)\ge n\left(\eta_4\left|\lambda_{s_n}\right|\right)+N\left(\left|\lambda_{s_n}\right|\right)\ge N\left(\eta_4\left|\lambda_{s_n}\right|\right).$$

This contradiction proves the theorem. ►

Example 3. *Condition (8) holds if* $\lim\limits_{n\to\infty} n/\left|\lambda_n\right|^\rho=\Delta\in(0;+\infty)$, *and also if* $|\lambda_n|=\varphi(n)(1+o(1))$ *as* $n\to\infty$, *where* φ *is an increasing, positive, and continuously differentiable function on* $[0;+\infty)$, *for which* $x\varphi'(x)/\varphi(x)\to+\infty$ *as* $x\to+\infty$, *because*

$$\left|\lambda_k/\lambda_n\right|=(1+o(1))\exp\left(\ln\varphi(\lambda_k)-\ln\varphi(\lambda_n)\right)=$$

$$=(1+o(1))\exp\left(-\int_k^n t\frac{\varphi'(t)}{\varphi(t)}dt\right)\le(1+o(1))\left(k/n\right)^{1/\rho}$$

for every $\rho>0$ *if* $k\to+\infty$ *and* $n\ge k$.

Theorem 5 ([1]). *Let* (λ_n) *be a sequence of distinct nonzero complex numbers such that*

$$(\exists\rho\in(0;1))(\exists n_0\in\mathbb{N})(\forall n\ge n_0):\left|\lambda_{n-1}/\lambda_n\right|\le\left((n-1)/n\right)^{1/\rho}. \qquad (10)$$

Then for an entire function $L(z)=\prod\limits_{n=1}^{\infty}(1-z/\lambda_n)$, *one has*

$$\ln\left|\lambda_n L'(\lambda_n)\right|\ge N\left(\left|\lambda_n\right|\right)-qn+O(1),\ n\to+\infty,$$

where

$$q:=\frac{1-\pi\rho\operatorname{ctg}(\pi\rho)}{\rho}.$$

Proof. Since [1]

$$\frac{1}{n}\left(\sum_{k=1}^{n-1}\ln\left(1-\left(k/n\right)^{1/\rho}\right)+\sum_{k=n+1}^{\infty}\ln\left(1-\left(n/k\right)^{1/\rho}\right)\right)\ge$$

$$\ge\int_0^1\ln\left(1-x^{1/\rho}\right)dx+\int_1^\infty\ln\left(1-\frac{1}{x^{1/\rho}}\right)dx=-q,$$

we have

$$\ln|\lambda_n L'(\lambda_n)|\ge N\left(\lambda_n\right)+\sum_{k=1}^{n-1}\ln\left(1-\left|\lambda_k\right|/\lambda_n\right)+\sum_{k=n+1}^{\infty}\ln\left(1-\left|\lambda_n\right|/\lambda_k\right)\ge$$

$$\ge N\left(\left|\lambda_n\right|\right)-qn-c_1,\ c_1>0,$$

which proves the inequality (10). ►

For other asymptotic properties of entire functions of finite order, see in [13, 23, 24, 28, 30-34, 45-47, 49, 51, 54].

4.8. Entire functions of regular growth. The detailed relationship between the asymptotic properties of an entire function and the properties of the sequence of its zeros is established in the theory of entire functions of completely regular growth (see [1, 13, 30, 31, 45-47, 49, 50]). The theory of entire functions of completely regular growth is presented in detail in [13, 30, 31, 45-47, 50].

An entire function f of order $\rho \in (0;+\infty)$ with the indicator h_f is said to be of completely regular growth in the sense of Levin-Pflüger, if there eaxists a set $E \subset [0;+\infty)$ of zero relative measure, i.e., an E_0-set, such that uniformly with respect to $\varphi \in [0;2\pi]$ (see [13, 30, 31, 45-47])

$$\ln\left| f(re^{i\varphi})\right| = r^{\rho}h_f(\varphi) + o(r^{\rho}),\ E_0 \not\ni r \to +\infty .$$

In this case, a set $E \subset [0;+\infty)$ is called a set of zero relative measure if

$$\lim_{r\to+\infty} \frac{\mu\left(E \cap [0;r]\right)}{r} = 0 ,$$

where μ is the Lebesgue linear measure on $\mathbb{R}$. We shall say [13, 30, 31, 45-47] that a sequence of complex numbers (λ_n) has an angular density of index ρ if, for almost all $\alpha \in \mathbb{R}$ and almost all $\beta \in \mathbb{R}$, $\alpha < \beta$, there exists a finite limit

$$\lim_{t\to+\infty} \frac{n(t;\alpha;\beta)}{t^{\rho}} = \Delta_0(\alpha;\beta),\ n(t;\alpha;\beta) := \sum_{|\lambda_n|\le t,\ \alpha<\arg\lambda_n\le\beta} 1 \cdot \tag{1}$$

The angular density of index ρ of a sequence (λ_n) is called [13, 30, 31, 45-47] a non-decreasing and left-continuous function $\Delta : \mathbb{R} \to \mathbb{R}$, which is determined, to within an additive constant, by the equality $\Delta(\beta) - \Delta(\alpha) = \Delta_0(\alpha;\beta)$, $\alpha < \beta$. If ρ is a noninteger, then a sequence (λ_n) having an angular density of index ρ is called regularly distributed [13, 30, 31, 45-47]. If ρ is an integer, then a sequence (λ_n) is called regularly distributed if it has an angular density of index ρ and there exists a finite limit [13, 30, 31, 45-47]:

$$\delta = \lim_{R\to+\infty} \sum_{|\lambda_n|\le R} \frac{1}{\lambda_n^{\rho}} .$$

We shall say that a set $U(a_n;\rho_n) \subset \mathbb{C}$, $U(a_n;\rho_n) = \left\{z : \left|z - a_n\right| \le \rho_n\right\}$, of disks has zero linear density if (see [13, 30, 31, 45-47])

$$\lim_{r\to+\infty} \frac{1}{r} \sum_{|a_n|\le r} \rho_n = 0 .$$

An C_0-set is called [13, 30, 31, 45-47] a set $C \subset \mathbb{C}$ that is contained within some collection of disks $U(a_n; \rho_n) \subset \mathbb{C}$ of zero linear density.

Theorem 1 (see [13, 30, 31, 45-47]). *Let f be an entire function of order $\rho \in (0; +\infty)$ with the indicator h_f. Then the following conditions are equivalent:*

1) there exists an exceptional C_0-set $C \subset \mathbb{C}$ such that

$$\ln|f(z)| = |z|^\rho h_f(\arg z) + o\left(|z|^\rho\right), \ C \not\ni z \to +\infty .$$

2) there exists an exceptional E_0-set $E \subset [0; +\infty)$ such that

$$\ln\left|f(re^{i\varphi})\right| = r^\rho h_f(\varphi) + o(r^\rho), \ E \not\ni r \to +\infty ,$$

holds uniformly in $\varphi \in [0; 2\pi]$;

3) there exists a sequence (r_k) such that $0 < r_k \uparrow +\infty$, $r_k / r_{k+1} \to 1$ as $k \to \infty$, and uniformly in $\varphi \in [0; 2\pi]$

$$\ln\left|f(r_k e^{i\varphi})\right| = r_k^\rho h_f(\varphi) + o(r_k^\rho), \ k \to \infty ,$$

4) the sequence (λ_k) of zeros of the function f is regularly distributed;

5) uniformly in $\varphi \in [0; 2\pi]$ one of the following conditions hold:

$$\lim_{r\to+\infty} r^{-\rho} \int_0^r \frac{\ln\left|f(te^{i\varphi})\right|}{t} dt = \frac{1}{\rho} h_f(\varphi), \ \lim_{r\to+\infty} r^{-\rho} \int_0^r \frac{dt}{t} \int_0^t \frac{\ln\left|f(ue^{i\varphi})\right|}{u} du = \frac{1}{\rho^2} h_f(\varphi) ;$$

6) for any $p \in [1; +\infty)$ holds

$$\lim_{r\to+\infty} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\ln\left|f(re^{i\varphi})\right|}{r^\rho} - h(\varphi) \right|^p d\varphi \right\}^{1/p} = 0, \ r \to +\infty;$$

7) for each $k \in \mathbb{Z}$ there exists a finite limit:

$$\lim_{r\to+\infty} \frac{C_k(r)}{r^\rho} = d_k , \ C_k(r) := \frac{1}{2\pi} \int_0^{2\pi} \ln\left|f(re^{i\varphi})\right| e^{-ik\varphi} d\varphi, \ k \in \mathbb{Z} .$$

In addition, if condition 2) is satisfied, then there exists a limit (1) for all $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$, except, perhaps, a some countable set of values of α and β for which $h_f'(\alpha+) \neq h_f'(\alpha-)$ and $h_f'(\beta+) \neq h_f'(\beta-)$. In this case,

$$\Delta_0(\alpha; \beta) = \frac{s_0(\alpha; \beta)}{2\pi\rho} ,$$

where

$$s_0(\alpha;\beta) = h'_f(\beta) - h'_f(\alpha) + \rho^2 \int_\alpha^\beta h_f(\varphi)d\varphi,$$

and the function $s(\varphi) = 2\pi\rho\Delta(\varphi)$ *is the associated measure of the function* h_f. *If condition 7) is true, then*

$$d_k = \frac{1}{2\pi}\int_0^{2\pi} h_f(\varphi)e^{-ik\varphi}d\varphi.$$

If the sequence (λ_n) is situated on a finite system of rays $\{z : \arg = \psi_j\}$, $j \in \{1,...,m\}$, $0 \le \psi_1 < \psi_2 < ... < \psi_m < 2\pi$, then condition (1) is equivalent to the condition [13, 30, 31, 45-47]

$$\lim_{t\to+\infty} \frac{n(t,\psi_j; f)}{t^\rho} = \Delta_j, \quad \Delta_j \in [0;+\infty), \quad j \in \{1,...,m\},$$

where $n(t,\psi_j; f) := \sum_{|\lambda_n|\le t,\, \arg\lambda_n = \psi_j} 1.$

More precise asymptotic estimates can be obtained for entire functions of improved regular growth, which have been studied in [6-9, 17-21, 25, 35-44, 56, 57]. We present here some fundamental facts of the theory of entire functions of improved regular growth.

An entire function f is said to be of improved regular growth [7, 35] if, for some $\rho \in (0;+\infty)$, $\rho_1 \in (0;\rho)$ and a 2π-periodic ρ-trigonometrically convex function $h \not\equiv -\infty$, there exists a set $U \subset \mathbb{C}$ contained in a union of disks with finite sum of radii such that

$$\ln|f(z)| = r^\rho h(\varphi) + o(r^{\rho_1}),\ U \not\ni z = re^{i\varphi} \to \infty.$$

If an entire function f is a function of improved regular growth, then it has [7, 35] the order ρ and the indicator $h(\varphi)$.

Let f be an entire function with $f(0) = 1$, let (λ_n) be a sequence of its zeros, let $p \le \rho$ is the least integer for which $\sum_{n\in\mathbb{N}} |\lambda_n|^{-p-1} < +\infty$ and let Q_ρ be the coefficient at z^ρ of an exponential factor in the Hadamard-Borel representation of an entire function f of order $\rho \in (0;+\infty)$.

Theorem 2 (see [6-9, 17-21, 35-44, 56, 57]). *Let* f *be an entire function of order* $\rho \in (0;+\infty)$ *with zeros on a finite system of rays*

$\{z:\arg=\psi_j\}$, $j\in\{1,\dots,m\}$, $0\le\psi_1<\psi_2<\dots<\psi_m<2\pi$. *Then the following assertions are equivalent:*

1) for some $\rho_2\in(0;\rho)$ *and each* $j\in\{1,\dots,m\}$ *holds*

$$n(t,\psi_j;f)=\Delta_j t^{\rho}+o(t^{\rho_2}),\ t\to+\infty,\ \Delta_j\in[0;+\infty),$$

and, in additon, for integer ρ *and some* $\rho_3\in(0;\rho)$ *and* $\delta_f\in\mathbb{C}$

$$\sum_{0<|\lambda_n|\le r}\lambda_n^{-\rho}=\delta_f+o(r^{\rho_3-\rho}),\ r\to+\infty;$$

2) f *is a function of improved regular growth with indicator* $h(\varphi)$. *Moreover, if* ρ *is a noninteger number, then*

$$h(\varphi)=\sum_{j=1}^{m}h_j(\varphi),$$

where $h_j(\varphi)$ *is a* 2π*-periodic function defined in the interval* $[\psi_j;\psi_j+2\pi)$ *by the equality*

$$h_j(\varphi)=\frac{\pi\Delta_j}{\sin\pi\rho}\cos\rho(\varphi-\psi_j-\pi).$$

For $\rho\in\mathbb{N}$, *we have*

$$h(\varphi)=\begin{cases}\tau_f\cos(\rho\varphi+\theta_f)+\sum_{j=1}^{m}h_j(\varphi), & p=\rho,\\ Q_\rho\cos\rho\varphi, & p=\rho+1,\end{cases}$$

where $\tau_f=|\delta_f/\rho+Q_\rho|$, $\theta_f=\arg(\delta_f/\rho+Q_\rho)$ *and* $h_j(\varphi)$ *is a* 2π*-periodic function such that on* $[\psi_j;\psi_j+2\pi)$

$$h_j(\varphi)=\Delta_j(\pi-\varphi+\psi_j)\sin\rho(\varphi-\psi_j)-\frac{\Delta_j}{\rho}\cos\rho(\varphi-\psi_j).$$

3) for some $\rho_1\in(0;\rho)$ *there exists a sequence* (r_k) *such that* $0<r_k\uparrow+\infty$, $r_{k+1}^{\rho}-r_k^{\rho}=o(r_k^{\rho_1})$ *as* $k\to\infty$, *and uniformly in* $\varphi\in[0;2\pi]$:

$$\ln|f(r_ke^{i\varphi})|=r_k^{\rho}h(\varphi)+o(r_k^{\rho_1}),\ k\to\infty;$$

4) for some $\rho_4\in(0;\rho)$ *the relation*

$$\int_1^r\frac{\ln|f(te^{i\varphi})|}{t}dt=\frac{r^{\rho}}{\rho}h(\varphi)+o(r^{\rho_4}),\ r\to+\infty,$$

holds uniformly in $\varphi\in[0;2\pi]$;

5) for some $\rho_5\in(0;\rho)$ *and* $k_0\in\mathbb{Z}$ *and each* $k\in\{k_0;k_0+1;\dots;k_0+m-1\}$,

$$C_k(r) = c_k r^{\rho} + o(r^{\rho_5}),\ r \to +\infty,$$

where

$$c_k := \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\varphi} h(\varphi)d\varphi = \frac{\rho}{\rho^2 - k^2}\sum_{j=1}^{m}\Delta_j e^{-ik\psi_j},$$

if $\rho \in (0;+\infty)\setminus\mathbb{N}$; *for* $\rho \in \mathbb{N}$ *the following relation is true:*

$$c_k = \begin{cases} \dfrac{\rho}{\rho^2 - k^2}\sum\limits_{j=1}^{m}\Delta_j e^{-ik\psi_j}, & |k| \neq \rho = p, \\ \dfrac{\tau_f e^{i\theta_f}}{2} - \dfrac{1}{4\rho}\sum\limits_{j=1}^{m}\Delta_j e^{-i\rho\psi_j}, & k = \rho = p, \\ 0, & |k| \neq \rho = p+1, \\ \dfrac{Q_\rho}{2}, & k = \rho = p+1, \end{cases}$$

6) for some $\rho_6 \in (0;\rho)$ *and every* $q \in [1;+\infty)$, *one has*

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\ln\left|f(re^{i\varphi})\right|}{r^{\rho}} - h(\varphi)\right|^q d\varphi\right\}^{1/q} = o(r^{\rho_6 - \rho}),\ r \to +\infty.$$

4.9. The Hardy space. Above, we discussed the factorization of entire functions. Similar problems can be considered for other classes of holomorphic functions. Let $1 \le p \le +\infty$ and $H_p(\mathbb{C}_+)$ be the space of functions holomorphic in the right-hand half-plane $\mathbb{C}_+ := \{z = x + iy : x > 0\}$, for which:

$$\|f\| := \sup\left\{\left|f(x+iy)\right| : x > 0\right\} < +\infty,\ p = +\infty,$$

$$\|f\|^p := \sup\left\{\int_{-\infty}^{+\infty}\left|f(x+iy)\right|^p dy : x > 0\right\} < +\infty,\ p \in [1;+\infty).$$

Theorem 1 ([26, 47, 52, 54]). *Every function* $f \in H_p(\mathbb{C}_+)$, $f \not\equiv 0$, $1 \le p \le +\infty$, *can be represented in the form*

$$f(z) = e^{ia_0 + a_1 z}\prod_{|\lambda_n|\le 1}\frac{z - \lambda_n}{z + \overline{\lambda_n}}\prod_{|\lambda_n|>1}\frac{1 - z/\lambda_n}{1 + z/\overline{\lambda_n}} \times$$

$$\times\exp\left\{\frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{(tz+i)}{(1+t^2)(t+iz)}\ln\left|f_0(t)\right|dt + \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{(tz+i)}{(1+t^2)(t+iz)}dh(t)\right\}, \quad (1)$$

where a_0 *and* a_1 *are real constants,* $a_1 \le 0$, $f_0 \in L_p(\mathbb{R})$ *is some function,* h *is a nonincreasing function on* $(-\infty;+\infty)$ *whose derivative is equal to zero almost everywhere,* (λ_n) *is the sequence of zeros of the function* f, *satisfying*

$$\sum_{n=1}^{\infty} \frac{\operatorname{Re}\lambda_n}{1+|\lambda_n|^2} < +\infty, \quad \int_{-\infty}^{+\infty} \frac{|\ln|f_0(t)||}{1+t^2} dt < +\infty, \quad \int_{-\infty}^{+\infty} \frac{1}{1+t^2} |dh(t)| < +\infty. \qquad (2)$$

Conversely, if the constants a_0 *and* a_1, *the functions* f_0 *and* h, *and the sequence* (λ_n), $\lambda_n \in \mathbb{C}_+$, *satisfy the above conditions, then the function* f, *defined by formula (1), belongs to the space* $H_p(\mathbb{C}_+)$.

Example 1. *The function* $f(z) = e^{-z}$ *belongs to* $H_\infty(\mathbb{C}_+)$, *because* $|f(z)| = |e^{-z}| \le 1$ *for* $z \in \mathbb{C}_+$.

Example 2. *The function* $f(z) = (1+z)^{-2}$ *belongs to* $H_1(\mathbb{C}_+)$, *because*

$$\int_{-\infty}^{+\infty} |f(x+iy)| dx = \int_{-\infty}^{+\infty} \frac{1}{(1+x)^2 + y^2} dy \le \int_{-\infty}^{+\infty} \frac{1}{1+y^2} dy = \pi, \; x > 0.$$

Theorem 2 (Paley-Wiener) ([26, 47, 52, 54]). *The space* $H_2(\mathbb{C}_+)$ *coincides with the class of functions* f *holomorphic in* $\mathbb{C}_+$ *and representable as follows*:

$$f(z) = \int_0^{+\infty} e^{-tz} q(t) dt, \quad \operatorname{Re} z > 0,$$

where $q \in L_2(0;+\infty)$.

4.10. The Fourier transform. The convolution of functions. The Fourier transform of a function $f : \mathbb{R} \to \mathbb{C}$ is defined as the function $\hat{f}(y)$, given by the formula [26, 47, 52, 54]:

$$\hat{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-iyt} dt. \qquad (1)$$

The operator F, which maps a function f to the function $\hat{f}$ by formula (1), is called the Fourier transform operator. Formula (1) can be rewritten as: $\hat{f} = F(f)$. The inverse Fourier transform of a function $\hat{f} : \mathbb{R} \to \mathbb{C}$ is defined as the function f, which is given by [26, 47, 52, 54]:

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\hat{f}(y)e^{iyx}dy\,. \tag{2}$$

The operator F^{-1}, which maps a function $\hat{f}$ to the function f using formula (2), is called the inverse Fourier transform operator. Formula (2) can be rewritten as: $f=F^{-1}(\hat{f})$. Under certain conditions, the Fourier integral formula holds [26, 47, 52, 54]:

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{ixy}\left(\int_{-\infty}^{+\infty}f(t)e^{-iyt}dt\right)dy\,. \tag{3}$$

The last formula can be rewritten as: $f=F^{-1}(F(f))$. The inverse Fourier transform of a function f will be denoted by $\check{f}$. It is obvious that

$$\check{f}(y)=\hat{f}(-y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(t)e^{iyt}dt\,.$$

Example 1. *If*

$$f(x)=\begin{cases}1, & |x|\le\sigma,\\ 0, & |x|>\sigma,\end{cases}$$

then

$$\hat{f}(y)=\frac{1}{\sqrt{2\pi}}\int_{-\sigma}^{\sigma}e^{-iyt}dt=\frac{e^{iy\sigma}-e^{-iy\sigma}}{iy\sqrt{2\pi}}=\begin{cases}\sqrt{2/\pi}\,\dfrac{\sin(\sigma y)}{y}, & y\ne 0,\\ \sigma\sqrt{2/\pi}, & y=0.\end{cases}$$

Example 2. *For every function $f\in L_1(\mathbb{R})$ and each $y_0\in\mathbb{R}$ holds $\hat{f}(y-y_0)=e^{iy_0t}f(t)$. Indeed,*

$$\hat{f}(y-y_0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(t)e^{-i(y-y_0)t}dt=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(t)e^{iy_0t}e^{-iyt}dt\,.$$

Example 3. *For every function $f\in L_1(\mathbb{R})$ and each $t_0\in\mathbb{R}$ holds $f(t-t_0)=e^{-it_0y}\hat{f}(t)$. Indeed,*

$$f(t-t_0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(t-t_0)e^{-iyt}dt=$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(u)e^{-iy(u+t_0)}du=e^{-iyt_0}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(u)e^{-iyu}du\,.$$

Example 4. *Let $f(x)=e^{-|x|}$. Then*

$$\hat{f}(y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-|t|}e^{-iyt}dt=\sqrt{\frac{2}{\pi}}\int_{-\infty}^{+\infty}e^{-|t|}\cos ytdt=\sqrt{\frac{2}{\pi}}\frac{1}{y^2+1}.$$

Example 5. *Let* $f(x)=\dfrac{1}{x^2+\alpha^2}$, $\alpha\in\mathbb{R}\setminus\{0\}$. *Then, by the residue theorem, we have*

$$\hat{f}(y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{1}{t^2+\alpha^2}e^{-iyt}dt=\frac{1}{\alpha}\sqrt{\frac{\pi}{2}}e^{-\alpha|y|}.$$

Example 6. *Let* $f(x)=e^{-\alpha x^2}$, $\alpha\in(0;+\infty)$. *Then, by Cauchy's theorem, we obtain*

$$\hat{f}(y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-\alpha t^2}e^{-iyt}dt=\frac{1}{\sqrt{2\pi}}\int_{-\infty-iy}^{+\infty+iy}e^{-\alpha\varsigma^2}e^{-iy\varsigma}d\varsigma=$$

$$=\frac{1}{\sqrt{2\pi}}e^{-y^2/4\alpha}\int_{-\infty}^{+\infty}e^{-\alpha\xi^2}d\xi=\frac{1}{\sqrt{2\alpha}}e^{-y^2/4\alpha},$$

because

$$\int_{-\infty}^{+\infty}e^{-\alpha\xi^2}d\xi=\sqrt{\frac{\pi}{\alpha}}.$$

Thus, $\hat{f}(y)=\lambda f(y)$ *if* $\lambda=1$ *and* $f(x)=e^{-x^2/2}$.

The convolution of two functions $f:\mathbb{R}\to\mathbb{C}$ and $\varphi:\mathbb{R}\to\mathbb{C}$ is defined as the function $f*\varphi$, given by the following formula [26, 47, 52, 54]:

$$f*\varphi(x)=\int_{-\infty}^{+\infty}f(x-\tau)\varphi(\tau)d\tau.$$

Example 7. *Let* $f(x)=x$ *and*

$$\varphi(x)=\begin{cases}1, & |x|\le 1,\\ 0, & |x|>1.\end{cases}$$

Then

$$f*\varphi(x)=\int_{-\infty}^{+\infty}f(x-\tau)\varphi(\tau)d\tau=\int_{-1}^{1}(x-\tau)d\tau=2x.$$

Example 8. *Let*

$$f(x)=\begin{cases}1, & x\ge 0,\\ 0, & x<0,\end{cases}\quad \varphi(x)=\begin{cases}1, & x\ge 0,\\ 0, & x<0.\end{cases}$$

Then

$$f*\varphi(x)=\int_0^{+\infty} f(x-\tau)d\tau=\int_0^x d\tau = x,\ x>0,$$

$$f*\varphi(x)=\int_0^{+\infty} f(x-\tau)d\tau=0,\ x\le 0.$$

The above formulas are valid under certain conditions.

Theorem 1. *If* $f\in L_1(\mathbb{R})$, *then the function* $\hat{f}$ *is continuous and bounded on* $\mathbb{R}$ *and* $\hat{f}(y)\to 0$ *as* $y\to\infty$.

Theorem 2. *If* $f\in L_1(\mathbb{R})$ *and* $\varphi\in L_1(\mathbb{R})$, *then* $f*\varphi=\sqrt{2\pi}\,\hat{f}\,\overset{\wedge}{\varphi}$.

Proof. Using the Fubini theorem, we obtain

$$\int_{-\infty}^{+\infty} f*\varphi(x)e^{-iyx}dx=\int_{-\infty}^{+\infty} f(t)dt\int_{-\infty}^{+\infty}\varphi(x-t)e^{-iyx}dx=\int_{-\infty}^{+\infty} f(t)e^{-iyt}dt\int_{-\infty}^{+\infty}\varphi(u)e^{-iyu}du.$$

This implies the required statement. ►

Theorem 3 ([26, 47, 52, 54]). *If the function* $f\in L_1(\mathbb{R})$ *is continuous on* $\mathbb{R}$ *and is of bounded variation on each interval* $[a;b]\subset\mathbb{R}$, *then at every point* $x\in\mathbb{R}$ *the Fourier integral formula (3) holds, where the outer integral is understood as a Cauchy principal value integral.*

By $S(\mathbb{R})$ or $\mathcal{J}(\mathbb{R})$, we denote the set of functions that have all derivatives on $\mathbb{R}$ and

$$(\forall n\in\mathbb{Z}_+)(\forall p\in\mathbb{Z}_+):\|f\|_{p,n}:=\sup\left\{|x|^p\left|f^{(n)}(x)\right|:x\in\mathbb{R}\right\}<+\infty.$$

The space $S(\mathbb{R})$ is called the space of rapidly decreasing functions. For example, the function $f(x)=e^{-x^2}$ belongs to $S(\mathbb{R})$.

Theorem 4 ([26, 47, 52, 54]). *The Fourier operator is a one-to-one mapping from* $S(\mathbb{R})$ *onto* $S(\mathbb{R})$. *For every function* $f\in S(\mathbb{R})$ *at each point* $x\in\mathbb{R}$, *the following dual formula*

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\hat{f}(y)e^{iyx}dx,$$

the integral Fourier formula (3) and Parseval's equality

$$\int_{-\infty}^{+\infty} |f(t)|^2\,dt = \int_{-\infty}^{+\infty} \left|\hat{f}(y)\right|^2 dy,$$

are valid.

If $f \in L_2(\mathbb{R})$, then the function $\varphi_y(t) = f(t)e^{-iyt}$ does not necessarily belong to $L_1(\mathbb{R})$. This will be the case, for example, if $f(t) = 1/\left(1+|t|\right)$. In this regard, the definition of the Fourier transform in the space $L_2(\mathbb{R})$ is introduced differently. The Fourier transform of a function $f \in L_2(\mathbb{R})$ or the L_2-Fourier transform of the function $f \in L_2(\mathbb{R})$ is called a function $\hat{f} \in L_2(\mathbb{R})$ such that

$$\lim_{a\to+\infty} \int_{-\infty}^{+\infty} \left|\hat{f}(y) - \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(t)e^{-iyt}dt\right|^2 dy = 0.$$

The inverse Fourier transform of a function $\hat{f} \in L_2(\mathbb{R})$ is called a function $f \in L_2(\mathbb{R})$, for which [26, 47, 52, 54]

$$\lim_{a\to+\infty} \int_{-\infty}^{+\infty} \left|f(x) - \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \hat{f}(y)e^{iyx}dy\right|^2 dx = 0.$$

Theorem 5 (Plancherel) ([26, 47, 52, 54]). *For every function $f \in L_2(\mathbb{R})$ there exists its L_2-Fourier transform and the Parseval equality holds:*

$$\int_{-\infty}^{+\infty} \left|\hat{f}(y)\right|^2 dy = \int_{-\infty}^{+\infty} |f(x)|^2\,dx.$$

4.11. Intuitive ideas about generalized functions. When studying physical processes, Dirac used a function δ that has the following properties [10, 11, 15, 55]:

1) $\delta(x) \geq 0$ for all $x \in \mathbb{R}$;

2) $\delta(x) = \begin{cases} 0, & x \neq 0, \\ +\infty, & x = 0; \end{cases}$

3) $\int_{-\infty}^{+\infty} \delta(x)dx = 1$.

In his studies of physical processes, Dirac arrived at conclusions that were consistent with experimental results. However, among integrable functions, there are none that satisfy conditions 1)-3). Therefore, such a function δ should be considered in a different sense. This can be done similarly to one of the methods used to introduce irrational numbers. Specifically, two fundamental on $\mathbb{Q}$ sequences (u_n) and (v_n) of rational numbers are called equivalent if $u_n - v_n \to 0$ as $n \to \infty$. This relation is an equivalence relation and partitions all fundamental sequences on $\mathbb{Q}$ into mutually disjoint classes. Each such a class is called a real number. This class u, i.e., the real number u, is uniquely determined by one of its elements, that is, by one of the sequences (u_n). This fact is denoted as $u = (u_n)$. The sum of two real numbers $u = (u_n)$ and $v = (v_n)$ is defined as the equivalence class $u + v$ containing the sequence $(u_n + v_n)$. From this definition, all known properties of the set of real numbers can be established. Each rational number u corresponds to the class containing the sequence $(u;u;...)$. Numbers that are not rational are called irrational or generalized elements of the set of rational numbers. For instance, the number $\sqrt{2}$, defined by the sequence of decimal approximations $\sqrt{2} = (1,4;1,41,...)$, belongs to this category. Thus, if we only know rational numbers, the number $\sqrt{2}$ is a generalized element of the set $\mathbb{Q}$ and a generalized solution of the equation $u^2 = 2$.

The theory of generalized functions, like the theory of real numbers, can be constructed in various ways. Let us first consider the approach proposed by R. Sikorski and J. Mikusiński. A sequence (f_k) of continuous functions $f_k : \mathbb{R} \to \mathbb{C}$ defined on $\mathbb{R}$ is called an MS-fundamental sequence [10, 11, 15, 55] if there exist a sequence (F_k) of functions $F_k : \mathbb{R} \to \mathbb{C}$ and an integer $n \geq 0$ such that: 1) $F_k^{(n)}(x) = f_k(x)$ for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$; 2) the sequence (F_k) converges uniformly on every compact of $\mathbb{R}$. Two MS-fundamental sequences (f_k) and (q_k) are called equivalent [10, 11, 15, 55] if there exist an integer $n \geq 0$ and sequences (F_k) and (Φ_k) such that: 3) $F_k^{(n)}(x) = f_k(x)$, $\Phi_k^{(n)}(x) = g_k(x)$ for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$; 4) the sequence $(F_k - \Phi_k)$ converges uniformly to zero on every compact of $\mathbb{R}$. This equivalence relation partitions all MS-fundamental sequences into mutually disjoint classes. Each such a class is called a generalized function. A generalized function f is uniquely determined by any sequence (f_k) belonging to the corresponding class, and this fact is denoted as: $f = (f_k)$. Every continuous function f is a generalized function because we can take $f_k = f$, $n = 0$ and $F_k = f_k$.

According to the Weierstrass theorem, for every continuous function $f : \mathbb{R} \to \mathbb{C}$ on $\mathbb{R}$, there exists a sequence of polynomials that converges uniformly to f on every compact of $\mathbb{R}$. Hence, for every MS-fundamental sequence (f_k), there exists an equivalent sequence (g_k) of infinitely differentiable functions. Therefore, when referring to a generalized function $f=(f_k)$, we assume that (f_k) is a sequence of infinitely differentiable functions. If the sequence (f_k) is MS-fundamental, then [10, 11, 15, 55] for every $n\in\mathbb{Z}$ the sequence $(f_k^{(n)})$ is also MS-fundamental. Here, $f^{(n)}=(f_k^{(n)})$ denotes the n-th derivative of the function f. Thus, every generalized function has derivatives of all orders. The product of two arbitrary generalized functions cannot be naturally defined in the space of generalized functions. This problem remains insufficiently studied. In practice, generalized functions are multiplied by assigning a specific meaning to the product in each particular case. However, the product of a generalized function and a sufficiently smooth function can be correctly defined within the space of generalized functions.

Let $C_0^{(\infty)}(\mathbb{R})$ be the set of all infinitely differentiable functions $\varphi:\mathbb{R}\to\mathbb{C}$ on $\mathbb{R}$ such that $\varphi(x)=0$ for all x outside some finite interval $[a;b]$. The integral of the product of a generalized function $f=(f_k)$ and a continuous function φ is defined as the limit [10, 11, 15, 55]:

$$\int_{-\infty}^{+\infty} f(x)\varphi(x)dx := \lim_{k\to\infty}\int_{-\infty}^{+\infty} f_k(x)\varphi(x)dx. \tag{1}$$

The limit (1) does not exist for every continuous function φ. However, if $\varphi\in C_0^{(\infty)}(\mathbb{R})$, the limit (1) exists because, for a suitable n, we have:

$$\int_{-\infty}^{+\infty} f_k(x)\varphi(x)dx = \int_{-\infty}^{+\infty} F_k(x)\varphi^{(n)}(x)dx,$$

and the sequence (F_k) converges uniformly on compact sets. The most important generalized function is the Dirac δ-function. It is a generalized function $\delta=(\delta_k)$ defined by an MS-fundamental sequence satisfying the following properties [10, 11, 15, 55]: a) $\delta_k(x)\ge 0$ for all $k\in\mathbb{N}$ and $x\in\mathbb{R}$; b) there exists a sequence (ε_k), $0<\varepsilon_k\to 0$, such that $\delta_k(x)=0$ for all $x\notin[-\varepsilon_k;\varepsilon_k]$; c) all δ_k are infinitely differentiable functions on $\mathbb{R}$; d) $\int_{-\infty}^{+\infty}\delta_k(x)dx=1$, $k\in\mathbb{N}$. A sequence satisfying properties a)-d) is called a δ-sequence.

An example of a δ-sequence [10, 11, 15, 55] is the sequence $\delta_k(x) = k\omega_1(kx)$, where $\omega_1(x) = c_1 \exp\left(-1/(1-x^2)\right)$ if $|x|<1$ and $\omega_1(x)=0$ if $|x| \geq 1$, and the constant c_1 is chosen such that $\int\limits_{-\infty}^{+\infty} \delta_k(x)dx = 1$. From conditions a)-d), it follows that $\lim\limits_{k\to\infty} \delta_k(x) = 0$, $x \neq 0$, and $\lim\limits_{k\to\infty} \delta_k(0) = +\infty$.

Theorem 1. *If $\delta = (\delta_n)$ is a δ-sequence, then for any continuous function $\varphi : \mathbb{R} \to \mathbb{C}$ on $\mathbb{R}$ holds*

$$\lim_{k\to\infty} \int\limits_{-\infty}^{+\infty} \delta_k(x)\varphi(x)dx = \varphi(0),$$

that is

$$\int\limits_{-\infty}^{+\infty} \delta(x)\varphi(x)dx = \varphi(0).$$

Proof. According to the mean value theorem, we have

$$\int\limits_{-\infty}^{+\infty} \delta_k(x)\varphi(x)dx = \int\limits_{-\varepsilon_k}^{\varepsilon_k} \delta_k(x)\varphi(x)dx =$$

$$= \varphi(\theta_k) \int\limits_{-\varepsilon_k}^{\varepsilon_k} \delta_k(x)dx \to \varphi(0),\ k\to\infty,\ -\varepsilon_k < \theta_k < \varepsilon_k. \blacktriangleright$$

Theorem 1 provides a way to give a certain meaning to equations

$$\int\limits_{-\infty}^{+\infty} \delta(x)\varphi(x)dx = \varphi(0),\ \int\limits_{-\infty}^{+\infty} \delta(x)dx = 1,$$

and to interpret the δ-function as a sequence $\left(\delta_k(x)\right)$ of continuous functions with certain properties: $\delta(x) \approx \delta_k(x)$ for large k. The interpretation of generalized functions provided above is useful for addressing a number of problems. At the same time, in many cases, a more convenient interpretation of generalized functions is the one proposed by L. Schwartz and S. Sobolev (see [10, 11, 15, 54]).

4.12. Space $C_0^{(\infty)}(\mathbb{R})$. Let $C^{(\infty)}(\mathbb{R})$ be the set of all functions $\varphi : \mathbb{R} \to \mathbb{C}$ that are infinitely differentiable on $\mathbb{R}$, and let $C_0^{(\infty)}(\mathbb{R})$ be the set of infinitely differentiable functions $\varphi : \mathbb{R} \to \mathbb{C}$ with compact support, i.e., those functions $\varphi \in C^{(\infty)}(\mathbb{R})$ that take the value zero outside some interval

$[a;b]\subset\mathbb{R}$ (this interval may depend on φ). An example of a function $\varphi\in C_0^{(\infty)}(\mathbb{R})$ is the so-called “bump function” [10, 11, 15, 55]:

$$\omega_1(x)=\begin{cases} c_1\exp\left(-1/(1-x^2)\right), & |x|<1, \\ 0, & |x|\ge 1, \end{cases}$$

where $c_1>0$ is a constant. Further, we assume that this constant is chosen so that

$$\int_{-\infty}^{+\infty}\omega_1(x)dx=1.$$

Let $\omega_\varepsilon(x)=\frac{1}{\varepsilon}\omega_1\left(\frac{x}{\varepsilon}\right)$, $\varepsilon>0$. Then [10, 11, 15, 55]:

1) $\omega_\varepsilon\in C_0^{(\infty)}(\mathbb{R})$,

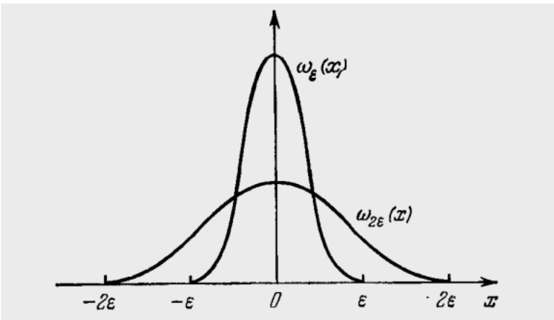
2) $\omega_\varepsilon(x)\ge 0$, $x\in\mathbb{R}$,

3) $\omega_\varepsilon(x)=\begin{cases} \frac{c_1}{\varepsilon}\exp\left(-\frac{\varepsilon^2}{\varepsilon^2-x^2}\right), & |x|<\varepsilon, \\ 0, & |x|\ge\varepsilon, \end{cases}$

4) $\int_{-\infty}^{+\infty}\omega_\varepsilon(x)dx=1$,

5) $\lim_{\varepsilon\to 0+}\omega_\varepsilon(x)=\begin{cases} 0, & x\ne 0, \\ +\infty, & x=0, \end{cases}$

6) $\varepsilon|\omega_\varepsilon(x)|\le c_1$, $x\in\mathbb{R}$, $\varepsilon\in(0;+\infty)$.



A function $\varphi:\mathbb{R}\to\mathbb{C}$ is said to have compact support [10, 11, 15, 55] if there exists an interval $[a;b]\subset\mathbb{R}$ such that $\varphi(x)=0$ for all $x\notin[a;b]$.

Hence, $C_0^{(\infty)}(\mathbb{R})$ is the set of all infinitely differentiable functions $\varphi:\mathbb{R}\to\mathbb{C}$ with compact support.

A sequence (φ_k) is called convergent [10, 11, 15, 55] on $C_0^{(\infty)}(\mathbb{R})$ to 0, if there exists an interval $[a;b]\in\mathbb{R}$ such that: a) $(\forall k\in\mathbb{N})(\forall x\in\mathbb{R}\setminus[a;b]):\varphi_k(x)=0$; b) for each $n\in\mathbb{Z}_+$ the sequence $\left(\varphi_k^{(n)}\right)$ uniformly converges on $[a;b]$ to zero, i.e.,

$$(\forall n\in\mathbb{Z}_+)(\forall\varepsilon>0)(\exists k^\bullet\in\mathbb{N})(\forall k\geq k^\bullet)(\forall x\in[a;b]):\left|\varphi_k^{(n)}(x)\right|<\varepsilon .$$

A sequence (φ_k) is said to converge [10, 11, 15, 55] in $C_0^{(\infty)}(\mathbb{R})$ to a function $\varphi\in C_0^{(\infty)}(\mathbb{R})$ if the sequence $\left(\varphi_k-\varphi\right)$ converges in $C_0^{(\infty)}(\mathbb{R})$ to zero. Hence, $\varphi_k\to\varphi$ in $C_0^{\infty}(\mathbb{R})$, if there exists an interval $[a;b]\subset\mathbb{R}$ such that: c) $(\forall x\in\mathbb{R}\setminus[a;b]):\varphi(x)=0$ and $(\forall k\in\mathbb{Z}_+)(\forall x\in\mathbb{R}\setminus[a;b]):\varphi_k(x)=0$; d) for each $n\in\mathbb{Z}_+$ the sequence $\left(\varphi_k^{(n)}\right)$ uniformly converges on $[a;b]$ to $\varphi^{(n)}$, that is

$$(\forall[a;b]\subset\mathbb{R})(\forall n\in\mathbb{Z}_+)(\forall\varepsilon>0)(\exists k^\bullet\in\mathbb{N})(\forall k\geq k^\bullet)(\forall x\in[a;b]):$$
$$\left|\varphi_k^{(n)}(x)-\varphi^{(n)}(x)\right|<\varepsilon .$$

Example 1. *The function* ω_1 *is a function with compact support.*

Example 2. *If* $\varphi\in C_0^{(\infty)}(\mathbb{R})$ *and* $\alpha\in C^{(\infty)}(\mathbb{R})$, *then* $\alpha\varphi\in C_0^{(\infty)}(\mathbb{R})$, *that is the space* $C_0^{(\infty)}(\mathbb{R})$ *is invariant under multiplication operator by infinitely differentiable function.*

Example 3. *The function* $\varphi(t)=\omega_1(t)\sin t$ *belongs to* $C_0^{(\infty)}(\mathbb{R})$.

Example 4 *A sequence* $\varphi_k(x)=\omega_1(x)+\omega_1(x)/k$ *converges on* $C_0^{(\infty)}(\mathbb{R})$ *to* $\omega_1(x)$.

4.13. Spaces of fundamental and generalized functions of one variable. A linear functional on a space $C_0^{(\infty)}(\mathbb{R})$ is called [10, 11, 15, 55] a function $f:C_0^{(\infty)}(\mathbb{R})\to\mathbb{C}$ such that for any $c_1\in\mathbb{C}$, $c_2\in\mathbb{C}$, $\varphi_1\in C_0^{(\infty)}(\mathbb{R})$ and $\varphi_2\in C_0^{(\infty)}(\mathbb{R})$ holds $f(c_1\varphi_1+c_2\varphi_2)=c_1 f(\varphi_1)+c_2 f(\varphi_2)$. A functional f is said to be continuous on $C_0^{(\infty)}(\mathbb{R})$ [10, 11, 15, 55] if, for any sequence (φ_k) that converges in $C_0^{\infty}(\mathbb{R})$, holds $\lim\limits_{k\to\infty} f(\varphi_k)=f\left(\lim\limits_{k\to\infty}\varphi_k\right)$. In order that a linear functional $f:C_0^{(\infty)}(\mathbb{R})\to\mathbb{C}$ be a continuous [10, 11, 15, 54], it is necessary and sufficient that for every compact set $K \subset \mathbb{R}$ there exist $c_1 \in (0;+\infty)$ and $\nu \in \mathbb{Z}_+$ such that $|f(\varphi)| \le c_1 \sum_{i \le \nu} \sup\left\{\left|\varphi^{(i)}(x)\right| : x \in K\right\}$ for all $\varphi \in C_0^{(\infty)}(\mathbb{R})$ that are equal to zero outside K.

The space $C_0^{(\infty)}(\mathbb{R})$ is called the space of test functions [10, 11, 15, 55], and it is often denoted by D. The elements of the space D are also called test functions.

A generalized function on $\mathbb{R}$ or a distribution on $\mathbb{R}$ is defined as any linear continuous functional on the space $C_0^{(\infty)}(\mathbb{R})$. The set of all generalized functions is called [10, 11, 15, 55] the space of generalized functions and it is typically denoted by D' or $(C_0^{(\infty)}(\mathbb{R}))'$. The value of a function $f \in D'$ on the element $\varphi \in D$ is often denoted by $(f;\varphi)$ or $\langle f;\varphi \rangle$. Therefore, $f(\varphi) = (f;\varphi) = \langle f;\varphi \rangle$. A generalized function is often denoted by $f(x)$, where $x \in \mathbb{R}$. However, in general, it is not possible to speak about the value of a generalized function f at a point $x \in \mathbb{R}$. This is because generalized functions $f(x)$ do not have pointwise values in the traditional sense.

Example 1. *The functional* $f(\varphi) = \varphi(0)$ *is linear and continuous, which means it is a generalized function.*

Example 2. *The functional* $f(\varphi) = \varphi(0) + 2\varphi''(0)$ *is linear and continuous, that is, it is a generalized function.*

Example 3. *The functional* $f(\varphi) = \varphi^2(0)$ *is not linear and therefore is not a generalized function.*

Example 4. *The functional* $f(\varphi) = \varphi(0) + 1$ *is not linear and therefore is not a generalized function.*

Example 5. *The functional* $f(\varphi) = \int_{-\infty}^{+\infty} x\varphi(x)dx$ *is linear and continuous, which means it is a generalized function. Integrating by parts, we conclude that it can also be represented in the form:*

$$f(\varphi) = \frac{1}{2!}\int_{-\infty}^{+\infty} x^2\varphi'(x)dx = \frac{1}{3!}\int_{-\infty}^{+\infty} x^3\varphi''(x)dx =$$

$$= \frac{1}{3}\int_{-\infty}^{+\infty} x\varphi(x)dx + \frac{1}{6}\int_{-\infty}^{+\infty} x^2\varphi'(x)dx + \frac{1}{18}\int_{-\infty}^{+\infty} x^3\varphi''(x)dx = \ldots .$$

. ***Example 6.*** *The functional*

$$(f;\varphi)=\int_{-\infty}^{+\infty} f_s(x)\varphi^{(s)}(x)dx$$

is a generalized function for every $f_s \in C(\mathbb{R})$.

Example 7. *Every function* $F:\mathbb{R}\to\mathbb{C}$ *of bounded variation within an interval* $[a;b]\subset\mathbb{R}$ *defines a generalized function* $f(\varphi)=\int_{-\infty}^{+\infty}\varphi(t)dF(t)$.

Example 8. *The functional* $f(\varphi)=\varphi^{(k)}(0)$ *is a generalized function.*

A generalized function $f\in(C_0^{(\infty)}(\mathbb{R}))'$ is called positive if $f(\varphi)\ge 0$ for all functions $\varphi\in C_0^{(\infty)}(\mathbb{R})$ such that $\varphi(x)\ge 0$ for $x\in\mathbb{R}$. For every positive generalized function $f\in(C_0^{(\infty)}(\mathbb{R}))'$ there exists a left-continuous and non-decreasing on $\mathbb{R}$ function $F:\mathbb{R}\to\mathbb{R}$ such that $f(\varphi)=\int_{-\infty}^{+\infty}\varphi(t)dF(t)$ for all $\varphi\in C_0^{(\infty)}(\mathbb{R})$, that is, every positive generalized function is a measure.

Example 9. *The functional*

$$f(\varphi)=\lim_{\varepsilon\to 0+}\int_{-\infty}^{+\infty}\frac{1}{x+i\varepsilon}\varphi(x)dx$$

is a generalized function denoted by $\frac{1}{x+i0}$.

4.14. Regular generalized functions of one variable. A function $\tilde{f}:\mathbb{R}\to\mathbb{C}$ is called locally summable or locally integrable on $\mathbb{R}$ [10, 11, 15, 55] if it is integrable on every finite interval $[a;b]\subset\mathbb{R}$. This fact we denote by $\tilde{f}\in L_{1,loc}(\mathbb{R})$. Every locally integrable on $\mathbb{R}$ function $\tilde{f}:\mathbb{R}\to\mathbb{C}$ generates a linear continuous functional on the space $C_0^{(\infty)}(\mathbb{R})$ (also denoted by $\tilde{f}$) by the formula [10, 11, 15, 55]:

$$(f;\varphi)=\int_{-\infty}^{+\infty}\tilde{f}(x)\varphi(x)dx. \qquad (1)$$

Thus, every locally integrable on $\mathbb{R}$ function $\tilde{f}$ is a generalized function [10, 11, 15, 55]. Consequently, the value of the functional $f\in(C_0^{(\infty)}(\mathbb{R}))'$ on the element $\varphi\in C_0^{(\infty)}(\mathbb{R})$ is also denoted by the symbol $\int_{-\infty}^{+\infty} f(x)\varphi(x)dx$. Thus,

$$f(\varphi)=(f;\varphi)=\int_{-\infty}^{+\infty} f(x)\varphi(x)dx .$$

In particular, each constant $c\in\mathbb{R}$ defines a constant generalized function (also denoted by c) and, we have [10, 11, 15, 55]:

$$(c;\varphi)=\int_{-\infty}^{+\infty} c\varphi(x)dx .$$

A generalized function f is called [10, 11, 15, 54] regular if it is generated by some locally summable on $\mathbb{R}$ function $\tilde{f}$ according to formula (1). A generalized function f is called singular if it is not regular.

Example 1. *The equality* $(f;\varphi)=\int_{-\infty}^{+\infty} e^{x}\varphi(x)dx$ *defines a generalized function.*

Example 2. *If* $f\in C^{(\infty)}(\mathbb{R})$ *and* $\varphi\in C_0^{(\infty)}(\mathbb{R})$, *then, integrating by parts, we obtain*

$$\int_{-\infty}^{+\infty} f'(x)\varphi(x)dx=-\int_{-\infty}^{+\infty} f(x)\varphi'(x)dx ,$$

$$\int_{-\infty}^{+\infty} f''(x)\varphi(x)dx=\int_{-\infty}^{+\infty} f(x)\varphi''(x)dx , ...,$$

$$\int_{-\infty}^{+\infty} f^{(k)}(x)\varphi(x)dx=(-1)^{k}\int_{-\infty}^{+\infty} f(x)\varphi^{(k)}(x)dx .$$

4.15. The Dirac δ-function and other singular functions. The functional δ, defined by the equality $(\delta;\varphi)=\varphi(0)$, is called [10, 11, 15, 55] the Dirac δ-function. Such a functional is linear and continuous, but is not generated by any locally summable function. Thus, the δ-function is a singular generalized function. The function δ_{x_0}, defined by the equality $(\delta_{x_0};\varphi)=\varphi(x_0)$, is called [10, 11, 15, 55] the δ-function concentrated at the point x_0. It is also denoted by $\delta(x-x_0)$. Therefore,

$$(\delta;\varphi)=\int_{-\infty}^{+\infty}\delta(x)\varphi(x)dx=\varphi(0) ,\ (\delta_{x_0};\varphi)=\int_{-\infty}^{+\infty}\delta(x-x_0)\varphi(x)dx=\varphi(x_0) .$$

The function $1/x$ is not locally summable on $\mathbb{R}$. However, it generates a linear continuous functional f on $C_0^{(\infty)}(\mathbb{R})$ by the equality

$$(f;\varphi)=v.p.\int_{-\infty}^{+\infty}\frac{1}{x}\varphi(x)dx. \qquad (1)$$

This generalized function is denoted by $\wp\frac{1}{x}$. In this case, we assume that $\varphi(x)=0$ for $x\notin[a;b]$. Moreover,

$$v.p.\int_{-\infty}^{+\infty}\frac{1}{x}\varphi(x)dx=\int_a^b\frac{\varphi(x)-\varphi(0)}{x}dx+v.p.\int_a^b\frac{\varphi(0)}{x}dx=$$

$$=\int_a^b\frac{\varphi(x)-\varphi(0)}{x}dx+v.p.\int_a^b\frac{\varphi(0)}{x}dx,$$

$$v.p.\int_a^b\frac{\varphi(0)}{x}\ \ dx=\lim_{\varepsilon\to0}\left(\int_a^{-\varepsilon}+\int_\varepsilon^b\ \right)\frac{\varphi(0)}{x}dx=\varphi(0)\ln\left|\frac{b}{a}\right|,$$

and the integral $\int_a^b x^{-1}(\varphi(x)-\varphi(0))dx$ exists as a Riemann integral. The function $1/x^2$ is not locally summable on $\mathbb{R}$. However, it also generates a generalized function $\wp_1\frac{1}{x^2}$ by the formula [10, 11, 15, 55]:

$$\left(\wp_1\frac{1}{x^2};\varphi\right)=v.p.\int_{-\infty}^{+\infty}\frac{\varphi(x)-\varphi(0)}{x^2}dx.$$

The product of two generalized functions cannot be defined naturally. The product of a generalized function $f\in(C_0^{(\infty)}(\mathbb{R}))'$ and a function $q\in C^{(\infty)}(\mathbb{R})$ is a generalized function $F=qf$ such that [10, 11, 15, 55]

$$\left(\forall\varphi\in C_0^{(\infty)}(\mathbb{R})\right):F(\varphi)=f(q\varphi).$$

Example 1. *If* $q\in C^{(\infty)}(\mathbb{R})$ *and* $q(0)=0$, *then* $q\delta=0$, *because* $(q\delta;\varphi)=\ (\delta;q\varphi)=q(0)\varphi(0)=0=(0;\varphi)$.

Example 2. *If* $q\in C^{\infty}(\mathbb{R})$ *and* $q(0)=1$, *then* $q\delta=\delta$, *because* $(q\delta;\varphi)=\ (\delta;q\varphi)=q(0)\varphi(0)=\varphi(0)=(\delta;\varphi)$.

Example 3. *Since*

$$\left(x\wp\frac{1}{x};\varphi\right)=\left(\wp\frac{1}{x};\varphi x\right)=v.p.\int_{-\infty}^{+\infty}\frac{x\varphi(x)}{x}dx=\int_{-\infty}^{+\infty}\varphi(x)dx=(1;\varphi),$$

we have $x\wp\frac{1}{x}=1$.

4.16. Derivative of a generalized function of one variable. The derivative of a generalized function $f \in (C_0^{(\infty)}(\mathbb{R}))'$ is called [10, 11, 15, 55] a generalized function $f' \in (C_0^{(\infty)}(\mathbb{R}))'$ such that

$$\left(\forall \varphi \in C_0^{\infty}(\mathbb{R})\right) : (f'; \varphi) = -(f; \varphi') . \qquad (1)$$

By definition, we have $f^{(0)} = f$. If $k \in \mathbb{N}$, then the k-th derivative of a generalized function $f \in (C_0^{(\infty)}(\mathbb{R}))'$ is called a generalized function $f^{(k)} \in (C_0^{(\infty)}(\mathbb{R}))'$ such that $(\forall \varphi \in C_0^{(\infty)}(\mathbb{R})) : (f^{(k)}; \varphi) = -(f^{(k-1)}; \varphi')$ (see [10, 11, 15, 55]).

Every generalized function has derivatives of all orders and the k-th derivative of a generalized function $f \in (C_0^{(\infty)}(\mathbb{R}))'$ is a generalized function $f^{(k)} \in (C_0^{(\infty)}(\mathbb{R}))'$ such that $(\forall \varphi \in C_0^{(\infty)}(\mathbb{R})) : (f^{(k)}; \varphi) = (-1)^{(k)} (f; \varphi^{(k)})$.

Remark 1. *The definition of the derivative of a generalized function by equality (1) is based on the fact that if the function f is continuously differentiable on $\mathbb{R}$ and $\varphi \in C_0^{(\infty)}(\mathbb{R})$, then*

$$\int_{-\infty}^{+\infty} f(t)\varphi'(t)dt = -\int_{-\infty}^{+\infty} f'(t)\varphi(t)dt .$$

Example 1. *Let*

$$\eta(x) = \begin{cases} 1, & x > 0, \\ 0, & x \le 0. \end{cases}$$

Then

$$(\eta'; \varphi) = -(\eta; \varphi') = \int_0^{+\infty} \eta(t)\varphi'(t)dt = \int_0^{+\infty} \varphi'(t)dt = \varphi(0) .$$

Hence, $\eta' = \delta$.

Example 2. *It is obvious that* $(\delta'; \varphi) = -(\delta; \varphi') = -\varphi'(0)$. *Therefore, the derivative of a* δ*-function is a generalized function* δ' *which corresponds the number* $-\varphi'(0)$ *to a function* $\varphi \in C_0^{(\infty)}(\mathbb{R})$.

Example 3. *Evidently,* $(\delta'(x - x_0); \varphi) = -(\delta(x - x_0); \varphi') = -\varphi'(x_0)$. *Therefore, the derivative of a function* $\delta(x - x_0)$ *is a generalized function* $\delta'(x - x_0)$ *which corresponds the number* $\varphi'(x_0)$ *to a function* $\varphi \in C_0^{(\infty)}(\mathbb{R})$.

Example 4. $(\delta^{(k)}; \varphi) = (-1)^k (\delta; \varphi^{(k)}) = (-1)^k \varphi^{(k)}(0)$.

Example 5. *If* $f(x) = \sin x$ *, then*

$$(\sin' x;\varphi) = -(\sin x;\varphi') = -\int_{-\infty}^{+\infty} \sin x \cdot \varphi'(x)dx =$$

$$= -\left(\sin x\varphi(x)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \cos x \cdot \varphi(x)dx \right) =$$

$$= \int_{-\infty}^{+\infty} \cos x \cdot \varphi(x)dx = (\cos x;\varphi) , \ \varphi \in C_0^\infty(\mathbb{R}) .$$

Thus, $\sin' = \cos$ *, in the sense of generalized functions. Similarly,*

$$(\cos' x;\varphi) = -(\cos x;\varphi') = -\int_{-\infty}^{+\infty} \cos x \cdot \varphi'(x)dx =$$

$$= -\left(\cos x\varphi(x)\Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} \sin x \cdot \varphi(x)dx \right) =$$

$$= -\int_{-\infty}^{+\infty} \sin x \cdot \varphi(x)dx = (-\sin x;\varphi) , \ \varphi \in C_0^\infty(\mathbb{R}) .$$

Hence, $\cos' = -\sin$ *, in the sense of generalized functions.*

Example 6. *We have the equality:*

$$\left(\left(\ln|x|\right)' ;\varphi \right) = -\left(\left(\ln|x|\right);\varphi' \right) = -\int_{-\infty}^{+\infty} \varphi'(x)\ln|x|\,dx =$$

$$= -\int_{0}^{+\infty} \varphi'(x)\ln x dx - \int_{-\infty}^{0} \varphi'(x)\ln(-x)dx =$$

$$= \int_{0}^{+\infty} \frac{\varphi(x)-\varphi(0)}{x}dx + \int_{-\infty}^{0} \frac{\varphi(x)-\varphi(0)}{x}dx =$$

$$= \int_{-\infty}^{+\infty} \frac{\varphi(x)-\varphi(0)}{x}dx = v.p.\int_{-\infty}^{+\infty} \frac{\varphi(x)}{x}dx = \left(\wp\frac{1}{x};\varphi \right), \ \varphi \in C_0^\infty(\mathbb{R}) .$$

Example 7. *Since*

$$\left(\left(\wp\frac{1}{x} \right)' ;\varphi \right) = -\left(\left(\wp\frac{1}{x} \right);\varphi' \right) = -\int_{-\infty}^{+\infty} \frac{\varphi'(x)-\varphi'(0)}{x}dx =$$

$$= -v.p\int_{-\infty}^{+\infty} \frac{\varphi(x)-\varphi(0)}{x^2}dx ,$$

we have

$$\left(\wp\frac{1}{x}\right)' = -\wp_1\frac{1}{x^2}.$$

Remark 2. *When considering generalized functions above, we taking into account as the main space* $D = C_0^{(\infty)}(\mathbb{R})$*, that is, we considered generalized functions from the class* $(C_0^{(\infty)}(\mathbb{R}))'$*. However, other spaces can also be taken as the space* D *, for example,* $C^{\infty}(\mathbb{R})$*,* $S(\mathbb{R})$*,* $C_0^{(k)}(\mathbb{R})$*, various spaces of holomorphic functions, and others.*

4.17. Harmonic and subharmonic functions. A function $u:\mathbb{R}^2 \to \mathbb{R}$ is called harmonic in a domain D if it has continuous second-order partial derivatives in D and satisfies at each point $(x;y)\in D$ the Laplace equation [3, 12, 26]:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\ (x;y)\in D.$$

The value $u(x;y)$ of a harmonic function u at a point $(x;y)$ is denoted as $u(z)$, and it is assumed that $z = x+iy$. The same property applies to subharmonic functions, which will be discussed further in this section.

The operator $\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ is called the Laplace operator [3, 12, 23, 26]. Hence, harmonic functions are precisely the functions that are solutions to the Laplace equation $\Delta u = 0$. If a function f is holomorphic in a domain D, then the function $u = \operatorname{Re} f$ is harmonic in D. Similarly, the function $u = \operatorname{Im} f$ is also harmonic in D. For a continuous function u in a domain D to be harmonic in D, it is necessary and sufficient that [3, 12, 23, 26]

$$u(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi} u\left(z+\rho e^{i\theta}\right)d\theta$$

for all $z\in D$ and ρ, $0<\rho<\inf\{|\varsigma - z| : \varsigma\in\partial D\}$. Moreover,

$$u(z)=\frac{1}{\pi\rho^2}\iint_{|\zeta - z|\le\rho} u(\zeta)\,d\xi d\eta\ ,\ \zeta=\xi+i\eta,$$

for every harmonic function u.

Example 1. *The function* $u(z)=e^x\cos y$ *is harmonic in* $\mathbb{R}^2$*, because*
$\frac{\partial^2 u}{\partial x^2}=e^x\cos y$ *and* $\frac{\partial^2 u}{\partial y^2}=-e^x\cos y$.

A function $w: D \to [-\infty;+\infty)$ is called [12, 23, 26, 46, 47] upper semicontinuous on a set D if $(\forall z \in D): \overline{\lim_{D \ni \zeta \to z}} w(\zeta) \le w(z)$. A function $w: \mathbb{C} \to [-\infty;+\infty)$ is called subharmonic in a domain $D \subset \mathbb{C}$ if it is a upper semicontinuous in D and for every $z \in D$ there exists r_0 such that for every $\rho \in (0;r_0)$ holds (see [12, 23, 26, 46, 47])

$$w(z) \le \frac{1}{2\pi\rho} \int\limits_{\partial U(z;\rho)} w(\zeta)|d\xi|, \qquad (1)$$

that is, $w(z) \le \frac{1}{2\pi} \int\limits_0^{2\pi} w(z+\rho e^{i\theta}) d\theta$. The integral (1) is either convergent or divergent to $-\infty$. The inequality (1) in this definition can be replaced by the inequality $w(z) \le \frac{1}{\pi\rho^2} \iint\limits_{U(z;\rho)} w(\zeta) d\xi d\eta$, where $\zeta = \xi + i\eta$, or the inequality

$$w(z) \le \frac{1}{\pi\rho^2} \int\limits_0^{\rho} \int\limits_0^{2\pi} w(z+\tau e^{i\theta}) \tau d\tau d\theta.$$

Let $C_0^\infty(D)$ be the set of all functions $\varphi: \mathbb{R}^2 \to \mathbb{C}$ infinitely differentiable in the domain D whose values are zero outside some compact set $E \subset D$, let $C_{0+}^\infty(D)$ be the set of non-negative functions $\varphi \in C_0^\infty(D)$. For a function $w: \mathbb{C} \to [-\infty;+\infty)$, $w \not\equiv -\infty$, to be subharmonic [12, 23, 26, 30, 46, 47] in the domain $D \subset \mathbb{C}$, it is necessary and sufficient that it be locally summable in D and

$$(\forall \varphi \in C_{0,+}^\infty(D)): \iint\limits_D w(z) \Delta\varphi(z) dxdy \ge 0. \qquad (2)$$

If the function w has continuous second-order partial derivatives in D, then the condition (2) is equivalent to the condition [12, 23, 26, 30, 46, 47]:

$$(\forall z = x+iy \in D): \Delta w(z) \ge 0. \qquad (3)$$

It is follows directly from the Green formula:

$$\iint\limits_D w(z) \Delta\varphi(z) dxdy = \iint\limits_D \Delta w(z) \varphi(z) dxdy. \qquad (4)$$

Thus, a function w is subharmonic in the domain D if Δw is a positive generalized function. Therefore, there exists a unique positive measure μ such that

$$(\forall \varphi \in C_0^\infty(D)): \iint\limits_D w(z) \Delta\varphi(z) dxdy = 2\pi \int\limits_D \varphi d\mu. \qquad (5)$$

This measure μ is called the Riesz measure of the function w. We can say that $\mu = \frac{1}{2\pi}\Delta w$ in the sense of generalized functions. The Riesz measure, interpreted as a function of sets, is defined on all Borel subsets $E \subset D$ and is finite on every compact set from D.

***Example 2* ([12, 23, 26, 45-47]).** *If the function w is harmonic in the domain D, then it is subharmonic in D.*

Example 3. *The function $w = e^x + y^2$ is subharmonic in $\mathbb{R}^2$, because* $\frac{\partial^2 w}{\partial x^2} = e^x$, $\frac{\partial^2 w}{\partial y^2} = 2$ *and* $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = e^x + 2 \geq 0$.

Example 4. *If the function $w \neq -\infty$ is subharmonic in a domain D, and $\eta : [-\infty; +\infty) \to [0; +\infty)$) is an increasing and convex function on an interval containing the range of w, then the function $\omega(z) = \eta(w(z))$ is subharmonic in D.*

Indeed, for twice continuously differentiable functions u and η, we have

$$\Delta\omega = \eta'\Delta u + ((\partial\omega / \partial x)^2 + (\partial\omega / \partial y)^2)\eta'' \geq 0.$$

This implies subharmonicity of ω. In the general case, we obtain the required conclusion from Jensen's inequality, according to which for any $z \in D$ and $r < r_z$:

$$\omega(z) = \eta\left(\frac{1}{2\pi r}\int_{|\zeta - z| = r} w(\zeta)\left|d\zeta\right|\right) \leq \frac{1}{2\pi r}\int_{|\zeta - z| = r} \omega(\zeta)\left|d\zeta\right|.$$

Example 5. *The function $h(z) = \ln|z|$ is subharmonic in $\mathbb{C}$ for all $\varphi \in C_0^\infty(\mathbb{C})$ and*

$$\iint_{(C)} \ln|z|\Delta\varphi(z)dxdy = \frac{1}{2\pi}\varphi(0) = \frac{1}{2\pi}\iint_{(C)} \varphi d\delta_0$$

is its Riesz measure, where $\mu = \delta_0$ is the Dirac measure centered at the point 0.

Indeed, subharmonicity of the function h is obvious, since h is harmonic in $\mathbb{C} \setminus \{0\}$ and $h(0) = -\infty$. Further, according to Green's formula,

$$\iint\limits_{\mathbb{C}} \ln|z|\Delta\varphi(z)dxdy = \left(\iint\limits_{|z|\le\varepsilon} + \iint\limits_{|z|>\varepsilon} \right) \ln|z|\Delta\varphi(z)dxdy =$$

$$= -\int\limits_{|z|=E} \left(\ln|z|\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial \ln|z|}{\partial n} \right)|dz| + o(1) = -\int\limits_{|z|=\mathrm{E}} \varphi\ln|z|\frac{\partial\varphi}{\partial n}|dz| + o(1) =$$

$$= \int\limits_{|z|=E} \left(-\varphi\frac{x}{|z|^2}\cos\alpha - \frac{y}{|z|^2}\varphi\cos\beta \right)|dz| + o(1) =$$

$$= \int\limits_{|z|=E} \left(\varphi\frac{x}{|z|^2}{}^{2} dx - \frac{y}{|z|^2}dy \right) + o(1) =$$

$$= \int\limits_{-\pi}^{\pi} \varphi(\varepsilon e^{i\theta})d\theta + 0(1) = 2\pi\varphi(0) + o(1)\,,\ \varepsilon \to 0+.$$

Since the left-hand side does not depend on ε, *we obtain the required conclusion from this.*

Example 6. *The function* $w(z) = |z|^\rho$, $0 < \rho < +\infty$, *is subharmonic in* $\mathbb{C}$ *and*

$$\mu(D) = \frac{\rho(\rho+1)}{\pi} \iint\limits_{(D)} |z|^{\rho-2}dxdy$$

is its Riesz measure. Indeed, if $\rho \ge 2$, *then the function* w *is a twice continuously differentiable in* $\mathbb{C}$ *and* $\Delta w = 2\rho(\rho+1)\Big|z\Big|^{\rho-2}$. *This implies the required statement. In the general case, we will use Green's formula. According to it, for all* $\varphi \in C_0^\infty(\mathbb{C})$ *holds*

$$\iint\limits_{\mathbb{C}} |z|^\rho \Delta\varphi(z)dxdy = \left(\iint\limits_{|z|\le\varepsilon} + \iint\limits_{|z|\ge\varepsilon} |z|^\rho \Delta\varphi(z)dxdy \right) =$$

$$= 2\rho(\rho+1) \iint\limits_{|z|\ge\varepsilon} |z|^{\rho-2}\varphi(z)dxdy + o(1) =$$

$$= 2\rho(\rho+1)\iint\limits_{\mathbb{C}} |z|^{\rho-2}\varphi(z)dxdy + o(1)\,,\ \varepsilon \to 0+,$$

which proves the required statement.

Example 7. *The function* $w(z) = r^\rho\cos\rho\varphi$, $0 < \rho < +\infty$, $z = re^{i\varphi}$, $|\varphi| \le \pi$, *is subharmonic in* $\mathbb{C}$ *and its Riesz measure is defined by the formula*

$$\mu(D)=\frac{\rho\cos\varphi\pi}{\rho}\int\limits_{D\cap(-\infty;0]}(-t)^{\rho-1}dt\,.$$

Indeed, since $w(z)=\mathrm{Re}(z^{\rho})$ *, the function* w *is harmonic in a domain* $D_{-}=\mathbb{C}\setminus\{z:\mathrm{Im}\,z=0,\mathrm{Re}\,z<0\}$. *Therefore, taking a sufficiently large* R, *by the Green formula, we obtain*

$$\iint\limits_{\mathbb{C}} w(z)\Delta\varphi(x)dxdy=\iint\limits_{D_0} w(z)\Delta\varphi(z)dxdy=$$

$$=\int\limits_{\partial(D_{-}\cap\{z:|z|\le R\})} w\frac{\partial\varphi}{\partial n}|dz|=2\int\limits_{-R}^{0}(-t)^{\rho}\cos(\varphi\pi)\,\varphi'(t)dt=$$

$$=2\rho\cos\pi\rho\int\limits_{-\infty}^{0}(-t)^{\rho-1}\varphi(t)dt\,,$$

whence it follows the required statement. Note that this function is continuous in $\mathbb{C}$, *but is not differentiable on the negative real ray.*

Example 8. *If* f *is a holomorphic function in the domain* D, *then the function* $w(z)=\ln|f(z)|$ *is subharmonic in* D *and its Riesz measure is determined by the equality*

$$\mu(D)=\sum_{\lambda_n\in D}1,$$

that is

$$\mu(D=)\sum_{\lambda_n\in D}\delta_{\lambda_n}\,,$$

where δ_{λ_n} *is the Dirac measure centered at the point* λ_n.

In fact, since $\ln|f(z)|$ *is a harmonic function in* $D\setminus\{\lambda_n\}$, *using Green's formula, we get*

$$\iint\limits_{(D)}\ln|f(z)|\Delta\varphi(z)dxdy=\sum_{\lambda_n\in D}\iint\limits_{|\lambda_n-z|\ge\varepsilon}\ln|f(z)|\Delta\varphi(z)dxdy+o(1)=$$

$$=-\sum_{\lambda_n\in D}\int\limits_{|\lambda_n-z|\ge\varepsilon}\varphi(z)\frac{\partial\ln|\lambda_n-z|}{\partial n}|dz|+o(1)=2\pi\sum_{\lambda_n\in D}\varphi(\lambda_n)+o(1)\,,\ \varepsilon\to 0+.$$

Example 9. *If* w *is a subharmonic function in* D, *then the function* w^{+} *is also subharmonic in* D.

In fact, $w^{+}(z)=\max\{w(z);0\}$, *and a maximum of two subharmonic functions is a subharmonic function.*

Example 10 **([12, 23, 26, 46, 47]).** *If* f *is a harmonic function in the domain* D*, then the functions* $w(z)=\ln|f(z)|$, $w(z)=\ln^{+}|f(z)|$ *ma* $|f(z)|^{\alpha}$, $\alpha>0$*, are subharmonic in* D.

Example 11 **([12, 23, 26, 46, 47]).** *If* u *is a harmonic function in the domain* D *and* $\alpha\geq 1$*, then* $|U(z)|^{\alpha}$ *is a subharmonic function in* D.

Example 12. *The function* $w(z)=\frac{1}{2}\ln\left(x^2+y^2\right)$ *is subharmonic in* $\mathbb{C}$*, because* $f(z)=z$ *is an entire function and*

$$w(z)=\frac{1}{2}\ln\left(x^2+y^2\right)=\ln|z|=\ln|f(z)|.$$

In this case, $\mu=\delta$ *is its Dirac measure, since using Green's formula, we obtain*

$$\iint_{\mathbb{C}}\ln|z|\Delta\varphi(z)dxdy=\iint_{\mathbb{C}}\varphi(z)\Delta\ln|z|dxdy=$$

$$=\left(\iint_{|z|\leq\varepsilon}+\iint_{|z|>\varepsilon}\right)\varphi(z)\Delta\ln|z|dxdy=$$

$$=\int_{-\pi}^{\pi}\varphi(\varepsilon e^{i\theta})d\theta+o(1)=2\pi\varphi(0)+o(1),\ \varepsilon\to 0+.$$

Example 13. *The function* $w(z)=|z|^{\rho}$, $0<\rho<+\infty$*, is subharmonic in* $\mathbb{C}$ *and*

$$\mu(D)=\frac{\rho(\rho+1)}{\pi}\iint_{(D)}|z|^{\rho-2}dxdy,$$

is its Riesz measure. Indeed, if $\rho\geq 2$*, then* w *is a twice continuously differentiable function in* $\mathbb{C}$ *and*

$$\Delta w=2\rho(\rho+1)|z|^{\rho-2}.$$

This yields the required statement. In general case, by using Green's formula, for all $\varphi\in C_0^{\infty}(\mathbb{C})$

$$\iint_{\mathbb{C}}|z|^{\rho}\Delta\varphi(z)dxdy=\iint_{\mathbb{C}}\varphi(z)\Delta|z|^{\rho}dxdy=$$

$$=\left(\iint_{|z|\leq\varepsilon}+\iint_{|z|\geq\varepsilon}\right)|z|^{\rho}\Delta\varphi(z)dxdy=$$

$$= 2\rho(\rho+1)\iint_{\mathbb{C}} |z|^{\rho-2}\varphi(z)dxdy + o(1),\ \varepsilon \to 0+.$$

Example 14 **([12, 23, 26, 45-47]).** *Let* $\rho \in (0;+\infty)$ *and let* $h:\mathbb{R}\to\mathbb{R}$ *is a* ρ*-trigonometrically convex and* 2π*-periodic function. Then* $w(z) = r^{\rho}h(\theta)$, $z = re^{i\theta}$, *is a subharmonic function in* $\mathbb{C}$. *In particular, the function* $w(z) = r^{\rho}\cos\rho\theta$ *is a subharmonic function in* $\mathbb{C}$.

Example 15 **([12, 23, 26, 45-47]).** *If* f *is an entire function, then* $w(z) = \ln|f(z)|$ *is a subharmonic function in* $\mathbb{C}$ *and* $\mu(D) = \sum_{\lambda_n \in D} 1$ *is its Riesz measure.*

Theorem 1. *Let* μ *is a positive measure in the domain* D, *which is defined on all Borel sets from* D *and is finite on every compact set from* D. *Then the function* $P(z) = \iint_{\overline{G}} \ln|z-\zeta|d\mu$ *is subharmonic in every bounded domain* G *such that* $\overline{G} \subset D$ *and its Riesz measure in* G *coincides with the restriction* μ *to* G.

Proof. Indeed,

$$\int_{(G)} P\Delta\varphi dxdy = \iint_{(G)}\left(\iint_{(G)} \Delta\varphi(z)dxdy\right)d\mu = \frac{1}{2\pi}\iint_{(G)} \varphi(\zeta)d\mu,\ \varphi \in C_0^{\infty}(G).\ \blacktriangleright$$

Theorem 2. *Let* $w \neq -\infty$ *be a subharmonic function in the domain* D *and* μ *be its Riesz measure in* D. *Then in every bounded domain* G *such that* $\overline{G} \subset D$, *the representation*

$$w(z) = P(z) + v(z), \tag{6}$$

is valid, where v *is a harmonic function in* G *and* $P(z) = \iint_{G} \ln|z-\zeta|d\mu$.

Proof. Let $v(z) = w(z) - P(z)$. Then

$$\iint_{(G)} v(z)\Delta\varphi(z)dxdy = 0,\ \varphi \in C_0^{\infty}(G).$$

Therefore, v is a harmonic function in G. For twice continuously differentiable functions v, this follows from Green's formula, and the general case is reduced to it. ►

Let us note that for a wide class of domains G, equality (6) can be rewritten as:

$$w(z) = -\iint_G g(z;\zeta)d\mu + u(z),$$

where g is the Green function of the domain G and

$$u(z) = \frac{1}{2\pi}\int_{\partial G} w\frac{\partial g}{\partial n}|d\zeta|.$$

Therefore, for a subharmonic in the domain D function $w \neq -\infty$, and for the corresponding class of domains G, $\overline{G} \subset D$, the following formula holds:

$$w(z) = -\iint_{(G)} g(\zeta;z)d\mu - \frac{1}{2\pi}\int_{\partial G} w\frac{\partial g}{\partial n}|d\zeta|, \qquad (7)$$

which is a generalization of the Poisson-Jensen formula for holomorphic functions. Using formula (7) for subharmonic functions in $\mathbb{C}$, one can obtain analogues of the theorems of Weierstrass, Borel, Hadamard, and others, which are well-known in the theory of entire functions. For more details, see [12, 23, 30, 46, 47].

Example 16. *The function* $w(z) = \ln|\sin z|$ *is a subharmonic function in* $\mathbb{C}$, $\mu(D) = \sum_{k\in D} 1$ *is its Riesz measure,*

$$P(z) = \int_{|\varsigma|\le 6,5} \ln|z-\varsigma|\,d\mu = \sum_{k\in\overline{-6;6}} \ln|z-k|$$

and

$$v(z) = \ln|\sin z| - P(z) = \ln\left|z\prod_{k=1}^{\infty}(1-(z/k\pi)^2)\right| - \sum_{k\in\overline{-6;6}}\ln|z-k| = \ln\left|\prod_{k=7}^{\infty}(1-(z/k\pi)^2)\right|.$$

4.18. Self-control questions.

1. Formulate the definition of the convergence exponent of a sequence.
2. Formulate the definition of the counting function of a sequence.
3. Formulate the definition of the averaged counting function of a sequence.
4. Formulate and prove theorems on the equivalent definition of the convergence exponent.
5. Formulate and prove a theorem on the connection between the convergence exponent and the genus of the canonical product.
6. Formulate the definition of the genus of an entire function.
7. Formulate and prove the Poincaré theorem on the genus of an entire function.
8. Formulate and prove the Hadamard-Borel theorem.
9. Formulate and prove theorems on the expansion of sine and cosine into infinite products.

10. Formulate the definition of the Fourier coefficients of an entire function.
11. Formulate and prove a theorem on finding the Fourier coefficients of an entire function.
12. Formulate and prove a theorem on the type of an entire function of noninteger order.
13. Formulate and prove the Lindelöf theorem on the type of an entire function of integer order.
14. Formulate the definition of the space $C_0^{(\infty)}(\mathbb{R})$.
15. Formulate the definition of a generalized function.
16. Formulate the definition of the space $(C_0^{(\infty)}(\mathbb{R}))'$.
17. Formulate the definition of the derivative of a generalized function.
18. Formulate the definition of a harmonic function.
19. Formulate the definition of a subharmonic function.
20. Formulate the definition of the Riesz measure.
21. Formulate the criterion for the subharmonicity of a function.

4.19. Exercises and problems.

1. Find the convergence exponent and the genus of a sequence:

1. $\lambda_k = k^2$.
2. $\lambda_k = k^2 \ln^4(1+k)$.
3. $\lambda_k = e^k$.
4. $\lambda_k = \ln^4(1+k)$.
5. $\lambda_k = k + \sqrt{k}$.
6. $\lambda_k = e^{\sqrt{k}} + k^4$.
7. $\lambda_k = e^k + k^4$.
8. $\lambda_k = \ln k$.
9. $\lambda_k = 2^k + \ln^4(1+k)$.
10. $\lambda_k = k^3 + k\ln^4(1+k)$.

2. (see [30, 31, 45-47, 51]) Prove that if $(\lambda_k) \subset \mathbb{C}_+$ is a sequence of complex numbers such that $|\lambda_k| \le 1$ and $\sum_k \operatorname{Re}\lambda_k < +\infty$, then the Blaschke product $B(z) = \prod_k (z-\lambda_k)/(z+\bar{\lambda}_k)$ uniformly and absolutely converges on every compact set in $\mathbb{C}_+$ and $|B(z)| \le 1$. If $(\lambda_k) \subset \mathbb{C}_+$ is a sequence of complex numbers such that $|\lambda_k|>1$ and $\sum_k \operatorname{Re}\lambda_k/|\lambda_k|^2<+\infty$, then the Blaschke product $B(z)=\prod_k (1-z/\lambda_k)/(1+z/\bar{\lambda}_k)$ has the same properties.

3. Let

$$f(z)=e^{Q(z)}\prod_{n=1}^{\infty}E\left(\frac{z}{\lambda_n};p\right)$$

be an entire function of order $\rho\in(0;+\infty)$, where $Q(z)=\sum_{k=0}^{\nu}Q_k z^k$ is a polynomial of degree $\nu\le\rho$, $p\le\rho$ is the genus of the sequence (λ_n) of its zeros and $E(w;p)$ is the Weierstrass primary factor. Prove the following assertions (see [13, 18,19, 23, 30, 31, 46, 51]):

1. $\alpha_k=Q_k$, $1\le k\le\nu$.

2. $\alpha_k=-\frac{1}{k}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^k}$, $k\ge\nu+1$; $\alpha_k=\begin{cases}Q_k, & k\in\overline{1;p},\\ -\sum_{n=1}^{\infty}\frac{1}{k\lambda_n^k}, & k\ge p+1.\end{cases}$

3. $C_k(R)=\frac{1}{2}Q_kR^k+\frac{1}{2k}\sum_{0<|\lambda_n|\le R}\left(\left(\frac{R}{\lambda_n}\right)^k-\left(\frac{\bar{\lambda}_n}{R}\right)^k\right)$, $1\le k\le\nu$.

4. $C_k(R)=-\frac{1}{2k}\sum_{|\lambda_n|>R}\left(\frac{R}{\lambda_n}\right)^k-\frac{1}{2k}\sum_{0<|\lambda_n|\le R}\left(\frac{\bar{\lambda}_n}{R}\right)^k$, $k\ge\nu+1$.

5. (see [13, 23, 30, 46]) If there exists a sequence (R_s) such that $0<R_s\uparrow+\infty$, $R_s/R_{s+1}\to 1$ as $s\to\infty$, and

$$(\forall k\in\mathbb{Z}):\lim_{s\to\infty}C_k(R_s)\big/R_s^{\rho}=c_k,$$

then

$$(\forall k\in\mathbb{Z}):\lim_{R\to\infty}C_k(R)\big/R^{\rho}=c_k.$$

6. (see [13, 16, 30, 31, 45-47, 51]) Let $\rho\in(0;+\infty)$. Then $\lim_{r\to+\infty}n(r)/r^{\rho}=\lim_{r\to+\infty}N(r)/(\rho r^{\rho})$ if one of these limits exists.

7. (see [6, 16, 19]) Let $\rho\in(0;+\infty)$ and $\Delta\in[0;+\infty)$. In order that for some $\rho_1\in(0;\rho)$ holds

$$n(r)=\Delta r^{\rho}+o(r^{\rho_1}),\ r\to+\infty, \tag{1}$$

it is necessary and sufficient that for some $\rho_2 \in (0;\rho)$

$$N(r) = \frac{\Delta}{\rho} r^{\rho} + o(r^{\rho_2}),\ r \to +\infty.$$

8. (see [6, 19]) Let $\Delta \in [0;+\infty)$. In order that for an entire function L of order $\rho \in (0;+\infty) \setminus \mathbb{N}$ there exists a system U of disks with finite sum of radii such that for some $\rho_3 \in (0;\rho)$ holds

$$\ln|L(z)| = \frac{\pi\Delta|z|^{\rho}}{\sin \pi\rho} \cos \rho(\varphi - \pi) + o\left(|z|^{\rho_3}\right),\ U \not\ni z = re^{i\varphi} \to \infty,$$

where $\varphi \in \operatorname{Arg} z$, it is necessary and sufficient that for some $\rho_1 \in (0;\rho)$ holds (1).

4. Let $L(z) = \prod_{n=1}^{\infty} \left(1 - z/\lambda_n\right)$ is an entire function of order $\rho < 1/2$. Prove the following assertions (see [1]):

1. If all $\lambda_n > 0$, then for every $\varphi \in [0;2\pi]$ there exists a sequence (r_k) such that $0 < r_k \uparrow +\infty$ as $k \to +\infty$, and

$$\left|L(r_k e^{i\varphi})\right| \geq M_L((1+o(1))(\cos \rho(\varphi - \pi))^{1/\rho} r_k), \quad k \to +\infty.$$

2. If all $\lambda_n > 0$, then there exists a sequence (r_k) such that $0 < r_k \uparrow +\infty$ as $k \to +\infty$, and

$$\left|L(r_k)\right| \geq M_L((1+o(1))(\cos \rho\pi)^{1/\rho} r_k), \quad k \to +\infty.$$

3. There exists a sequence (r_k) such that $0 < r_k \uparrow +\infty$ as $k \to +\infty$, and

$$m_L(r_k) \geq M_L((1+o(1))(\cos \rho\pi)^{1/\rho} r_k),\ k \to +\infty,$$

where $m_L(r) = \min\{|L(z)| : |z| = r\}$.

4. If $\rho = 0$, then there exists a sequence (r_k) such that $0 < r_k \uparrow +\infty$ and $m_L(r_k) = M_L((1+o(1))r_k)$ as $k \to +\infty$.

5. If

$$(\forall \rho > 0)(\exists \rho_1 > \rho)(\exists \alpha > 1)(\exists k_0)(\forall n \geq k) : |\lambda_k / \lambda_n| \leq (\alpha k / n)^{1/\rho_1} \qquad (2)$$

and $q = (1 - \pi\rho \operatorname{ctg}(\pi\rho))/\rho$, then there exists a C_0-set of disks of zero linear density such that

$$\ln|L(z)| \geq N(|z|) - q(1+o(1))n(|z|),\ C_0 \not\ni z \to \infty.$$

6. If all $\lambda_n > 0$ and the inequality (2) is true, then for each $\varphi_0 \in (0;\pi)$ holds

$$\left|L(re^{i\varphi})\right| \geq M_L((1+o(1))(\cos \rho(\varphi - \pi))^{1/\rho} r), \quad r \to +\infty,$$

uniformly in $\varphi \in [\varphi_0; 2\pi - \varphi_0]$.

7. If all $\lambda_n > 0$ and the inequality (2) is true, then there exists a sequence (r_k) such that $0 < r_k \uparrow +\infty$, $r_k / r_{k+1} \to 1$ as $k \to +\infty$ and uniformly in $\varphi \in [0; 2\pi]$

$$\left|L(r_k e^{i\varphi})\right| \geq M_L((1+o(1))(\cos \rho(\varphi - \pi))^{1/\rho} r_k), \quad k \to +\infty .$$

8. If all $\lambda_n > 0$, then the following conditions are equivalent: a) the inequality (2) is valid; b) there exists a C_0-set of disks of zero linear density such that $\left|L(z)\right| = M_L((1+o(1))\left|z\right|)$ as $C_0 \not\ni z \to \infty$; c) there exists a sequence (r_k) such that $0 < r_k \uparrow +\infty$, $r_k / r_{k+1} \to 1$ and $m_L(r_k) = M_L((1+o(1))r_k)$ as $k \to +\infty$; d) $\ln M_L(r) = N((1+o(1))r)$ as $r \to +\infty$.

9. If the inequality (2) holds for $0 < \rho < 1/2$, then the series

$$\sum_{n=1}^{\infty} \exp(N(r\left|\lambda_n\right|) / \left|\lambda_n L'(\lambda_n)\right|$$

converges for $r \in [0; R_1)$, where $R_1 = (\pi\rho \operatorname{ctg}(\pi\rho))^{1/\rho}$.

10. If $\left|\lambda_{n-1} / \lambda_n\right| \leq \Delta < 1$ for all $n > 1$, then $\ln M_L(r) = N(r) + O(1)$ as $r \to +\infty$.

5. Find the Fourier transform of a function:

1. $f(x) = \begin{cases} -1, & x \in [-2; 1], \\ 0, & x \notin [-2; 1]. \end{cases}$

2. $f(x) = \begin{cases} x, & x \in [0; 1], \\ 0, & x \notin [0; 1]. \end{cases}$

3. $f(x) = \begin{cases} x^2, & x \in [0; 1], \\ 0, & x \notin [0; 1]. \end{cases}$

4. Find the Fourier transform of a function f if $f(x) = \varphi(x)$ for $x \in \left[-\pi, \pi\right]$ and $f(x) = 0$ for $x \notin \left[-\pi; \pi\right]$, where φ is a solution of the differential equation $\varphi'' + \varphi = 0$, satisfying the initial conditions: $\varphi(0) = 0, \varphi'(0) = 1$.

5. Find the Fourier transform of a function f if $f(x) = \varphi(x)$ for $x \in \left[0, 1\right]$ and $f(x) = 0$ for $x \notin \left[0; 1\right]$, where φ is a solution of the differential equation $\varphi'' - \varphi = 0$, satisfying the initial conditions: $\varphi(0) = 1, \varphi'(0) = 0$.

6. Find the Fourier transform of a function f if $f(x)=\varphi(x)$ for $x\in[-1,0]$ and $f(x)=0$ for $x\notin[-1,0]$, where φ is a solution of the differential equation $\varphi''-4\varphi=0$, satisfying the initial conditions: $\varphi(0)=0, \varphi'(0)=1$.

7. Find the Fourier transform of a function f if $f(x)=\varphi(x)$ for $x\in[-1,0]$ and $f(x)=0$ for $x\notin[-1,0]$, where φ is a solution of the differential equation $\varphi''-\varphi'=0$, satisfying the initial conditions: $\varphi(0)=1$, $\varphi'(0)=0$.

6. Find the convolution $f*\varphi$ of the given functions:

1. $f(x)=x$, $\varphi(x)=e^x$.
2. $f(x)=\sin x$, $\varphi(x)=\operatorname{ch} x$.
3. $f(x)=x^3$, $\varphi(x)=x^2$.
4. $f(x)=\begin{cases} e^{-x}, & x\ge 0, \\ 0, & x<0, \end{cases}$ $\varphi(x)=\begin{cases} 1, & x\ge 0, \\ 0, & x<0. \end{cases}$
5. $f(x)=\begin{cases} \dfrac{1}{1+x^2}, & x\ge 0, \\ 0, & x<0, \end{cases}$ $\varphi(x)=\begin{cases} 1, & x\ge 0, \\ 0, & x<0. \end{cases}$
6. $f(x)=\begin{cases} 0, & x\ge 0, \\ e^{2x}, & x<0, \end{cases}$ $\varphi(x)=\begin{cases} 1, & x\ge 0, \\ 0, & x<0. \end{cases}$

7. Determine whether the given equality defines a generalized function $f\in(C_0^{(\infty)}(\mathbb{R}))'$:

1. $(f;\varphi)=2\varphi'(1)$.
2. $(f;\varphi)=2\varphi'(1)+\varphi(0)$.
3. $(f;\varphi)=\varphi(1)+1$.
4. $(f;\varphi)=\varphi^2(0)$.
5. $(f;\varphi)=e^{\varphi(0)}$.
6. $(f;\varphi)=\int_0^1(\varphi(x)+1)dx$.
7. $(f;\varphi)=\int_0^1\varphi(x+1)dx$.
8. $(f;\varphi)=\int_0^1\varphi(x^2)dx$.

9. $(f;\varphi)=\int_0^1 \varphi^2(x)dx$.

10. $(f;\varphi)=\int_0^1 \varphi(e^x)dx$.

11. $(f;\varphi)=\int_0^1 e^{\varphi(x)}dx$.

8. Using the definition of the generalized derivative, prove the following formulas:

1. $(x^2)'=2x$.
2. $(e^x)'=e^x$.
3. $(\sin x)'=\cos x$.
4. $(\cos x)'=-\sin x$.
5. $(x^3)'=3x^2$.
6. $(\sin 2x)'=2\cos 2x$.
7. $(\cos 2x)'=-2\sin 2x$.
8. $(x^4)'=4x^3$.
9. $(e^{2x})'=2e^{2x}$.
10. $(e^{-4x})'=-4e^{4x}$.
11. $(e^{5x})'=5e^{5x}$.

9. Find the derivative of the given generalized function:

1. $f(x)=\delta(x)\sin 2x$.
2. $f(x)=\delta'(x)\cos^2 x$.
3. $f(x)=\delta(x)e^{-x}$.
4. $f(x)=\delta(x)e^{x^2}$.
5. $f(x)=\delta(x)e^{-x^2}$.
6. $(f;\varphi)=\int_0^1 \varphi(x)dx$.
7. $(f;\varphi)=\int_0^1 \varphi(x)\sin x dx$.

8. $(f;\varphi)=\int_0^1 \varphi(x)\cos x dx$.

9. $(f;\varphi)=\int_0^1 \varphi(x)\cos x^2 dx$.

10. $(f;\varphi)=\int_0^1 \varphi(x)e^{-x^2} dx$.

10. Determine whether a function u is harmonic:

1. $u=x^2-y^2$.
2. $u=x^2+y^2$.
3. $u=x^4y+xy^4$.
4. $u=x^4+xy^2$.
5. $u=e^x\cos y$.
6. $u=\dfrac{x^2-y^2}{x^2+y^2}$.
7. $u=\dfrac{1}{2}\ln(x^2+y^2)$.
8. $u=e^x\sin y$.
9. $u=\arccos\dfrac{x}{y}$.
10. $u=\mathrm{arctg}\dfrac{y}{x}$.

11. Determine whether a function w is subharmonic:

1. $w=e^x+e^{2y}$.
2. $w=3x^4+4y^6$.
3. $w=(x^2+y^2)^{3/2}$.
4. $w=x^4+2x^2y^2+y^4$.
5. $w=x^2+y^2+e^x\cos y$.
6. $w=x^4+2x^2y^2+y^4+y^4e^{2x}$.
7. $w=\ln(x^2+y^2)+e^{2x}$.
8. $w=x^2+y^2+x^2e^y$.

9. $w = e^x \cos y + x^4$.

10. $w = e^x \sin y + y^6$.

4.20. Individual tasks.

1. Find the order of an entire function and determine whether it has a finite type:

1. $f(z) = \prod_{k=1}^{\infty}\left(1 - \frac{z}{k^3}\right)$.

2. $f(z) = z^3 e^{-2z+5} \prod_{k=2}^{\infty}\left(1 - \frac{z}{k \ln^2 k}\right)$.

3. $f(z) = \prod_{k=1}^{\infty}\left(1 - \frac{4z^2}{k^2}\right)$.

4. $f(z) = z^3 e^{-2z^2+z+2} \prod_{k=2}^{\infty}\left(1 - \frac{z}{k \ln^2 k}\right)$.

5. $f(z) = e^z \prod_{k=1}^{\infty}\left(1 - \frac{z}{k \ln^2(1+k)}\right)$

6. $f(z) = e^z \prod_{k=1}^{\infty}\left(1 - \frac{z}{k}\right) e^{\frac{z}{k}}$.

7. $f(z) = z \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2 \pi^2}\right)$.

8. $f(z) = e^{1+z+2z^2} \prod_{k=1}^{\infty}\left(1 - \frac{z}{k^2}\right)$.

9. $f(z) = e^{1+10z+2z^4} \prod_{k=1}^{\infty}\left(1 - \frac{z}{e^{\sqrt{k}}}\right)$.

10. $f(z) = (z-1)^5 e^{-2z^2+3} \prod_{k=2}^{\infty}\left(1 - \frac{z^2}{k \ln^2 k}\right)$.

11. $f(z) = z \prod_{k=2}^{\infty}\left(1 - \frac{z}{k^2 \ln k}\right)$.

12. $f(z) = \prod_{k=1}^{\infty}\left(1 + \frac{z}{e^k}\right)$.

13. $f(z) = \prod_{k=1}^{\infty}\left(1 + \frac{z}{k^\sigma}\right)$, $\sigma > 0$.

14. $f(z)=e^{z}\prod_{k=1}^{\infty}\left(1-\frac{z}{\sqrt{k}}\right)e^{\frac{z}{\sqrt{k}}+\frac{z^2}{2k}}.$

15. $f(z)=e^{z}\prod_{k=1}^{\infty}\left(1-\frac{zk^3}{2^k}\right).$

16. $f(z)=\prod_{k=1}^{\infty}\left(1+\frac{z}{k^2}\right)$

17. $f(z)=e^{1+10z+2z^4}\prod_{k=1}^{\infty}\left(1-\frac{ze^{\sqrt[3]{k}}}{2^k}\right).$

18. $f(z)=e^{z+3z^2}\prod_{k=1}^{\infty}\left(1-\frac{z}{\sqrt{k}\ln(1+k)}\right)e^{\frac{z}{\sqrt{k}\ln(1+k)}+\frac{z^2}{2k\ln^2(1+k)}}.$

19. $f(z)=e^{z}\prod_{k\in\mathbb{N}}E\left(\frac{z}{\lambda_k};2\right),\quad \lambda_{4k}=k^{1/2},\quad \lambda_{4k+1}=ik^{1/2},\quad \lambda_{4k+2}=-k^{1/2},$

$\lambda_{4k+3}=-ik^{1/2}.$

20. $f(z)=z\prod_{k=1}^{\infty}\left(1+\frac{z}{k^2}\right).$

21. $f(z)=e^{z}\prod_{k=1}^{\infty}\left(1-\frac{z}{k^2}\right).$

22. $f(z)=\prod_{k=2}^{\infty}\left(1-\frac{z}{k^2\ln k}\right).$

23. $f(z)=e^{2z}\prod_{k=1}^{\infty}\left(1-\frac{z}{k^3}\right).$

24. $f(z)=\prod_{k=1}^{\infty}\left(1-\left(\frac{2z}{(2k+1)\pi}\right)^2\right).$

25. $f(z)=z\prod_{k=1}^{\infty}\left(1+\frac{z^2}{k^2\pi^2}\right).$

26. $f(z)=\prod_{k=0}^{\infty}\left(1+\left(\frac{2z}{(2k+1)\pi}\right)^2\right).$

27. $f(z)=ze^{\frac{z}{2}}\prod_{k=1}^{\infty}\left(1+\frac{z^2}{4k^2\pi^2}\right)$.

28. $f(z)=z^2\prod_{k=1}^{\infty}\left(1+\frac{z^4}{4k^4\pi^4}\right)$.

29. $f(z)=(a-b)ze^{\frac{(a+b)z}{2}}\prod_{k=1}^{\infty}\left(1+\frac{(a-b)^2z^2}{4k^2\pi^2}\right)$.

30. $f(z)=\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right)$.

2. Represent the entire function as an infinite product:

1. $f(z)=e^{2z}-e^{z}$.
2. $f(z)=\cos z$.
3. $f(z)=\sin z$.
4. $f(z)=\operatorname{sh} z$.
5. $f(z)=\operatorname{ch} z$.
6. $f(z)=e^{z}-1$.
7. $f(z)=e^{az}-e^{bz}$.
8. $f(z)=\operatorname{ch} z-\cos z$.
9. $f(z)=e^{z-1}$.
10. $f(z)=\sin(z^2)$.
11. $f(z)=\cos\sqrt{z}$.
12. $f(z)=\frac{\sin(\pi\sqrt{z})}{\pi\sqrt{z}}$.
13. $f(z)=\frac{\sin(\pi z)}{z}$.
14. $f(z)=\sin(\pi z)$.
15. $f(z)=e^{2z+1}$.
16. $f(z)=\cos(\pi z)$.
17. $f(z)=\sin\sqrt{z}$.
18. $f(z)=\cos(z^2)$.
19. $f(z)=\sin(1-z)$.

20. $f(z)=\dfrac{\operatorname{sh} z}{z}$.

21. $f(z)=\dfrac{\sin\sqrt{z}}{\sqrt{z}}$.

22. $f(z)=e^{z}-e^{a}$.

23. $f(z)=\cos(\pi z)-\cos(\pi a)$.

24. $f(z)=e^{z^2}+e^{2z-1}$.

25. $f(z)=\dfrac{\cos z-\cos a}{1-\cos a}$.

26. $f(z)=\sin(z-a)+\sin a$.

27. $f(z)=e^{z-1}-e^{z^2}$.

28. $f(z)=\operatorname{sh}(z+1)$.

29. $f(z)=\operatorname{ch}(z-1)$.

30. $f(z)=\dfrac{\sin(\pi z)}{\pi z}$.

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