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Two-sided FD-method for nonlinear differential equations

Lazurchak I. I.

informatyka@drohobych.net

State Pedagogical University Drohobych, 3, Stryjska str., Drohobych, 82100, Ukraine,

A propose a new functional-discrete method for solving nonlinear differential equations. The method possesses the exponential convergence rate and can provide two-sided approximations. The user can control the exponential convergence through an embedded control mechanism.

1. Introduction

In the paper a control mechanism which guarantees the exponential convergence is developed for a wide class of nonlinearities independent of the generalized Lipschitz constant L (i.e. global). The idea of such approach for eigenvalue problems was recently announced in [20, 17].

Let us remind of the idea of Adomian decomposition method (ADM) [2] which can be also interpreted as the FD-method proposed in [19] for the Sturm-Liouville problems and is very close to the homotopy perturbation methods.

If we have to solve the operator equation

$$\hat{u} = -N(\hat{u}) + F, \tag{1}$$

then we can imbed it into the family of equations

$$u(t) = -tN(u(t)) + F, \quad t \in [0,1] \tag{2}$$

and obtain obviously $u(1) \equiv \hat{u}$.

We look for the solution of (2) in the form

$$u(t) = \sum_{j=0}^{\infty} t^j u^{(j)}, \tag{3}$$

and represent

$$N\Big(\sum_{j=0}^{\infty} t^j u^{(j)}\Big) = \sum_{j=0}^{\infty} t^j A_j, \tag{4}$$

where

$$A_j = \frac{1}{j!}\frac{\partial^j N(\sum_{k=0}^{\infty} t^k u^{(k)})}{\partial t^j}\bigg|_{t=0}. \tag{5}$$

Substituting (3) into (1) we have

$$\sum_{j=0}^{\infty} t^j u^{(j)} = -tN\Big(\sum_{j=0}^{\infty} t^j u^{(j)}\Big) + F. \tag{6}$$

Applying to this equality successively the operator $\frac{1}{(j+1)!}\frac{d^{j+1}}{dt^{j+1}}$ and then setting $t = 0$ we obtain the following recurrence formulas

$$\begin{aligned} &u^{(j+1)} = -A_j\left(N; u^{(0)}, \dots, u^{(j)}\right), \quad j = 0, 1, \dots \\ &A_0\left(N; u^{(0)}\right) = N\left(u^{(0)}\right), \quad u^{(0)} = F. \end{aligned} \tag{7}$$

Here $A_j\left(N; u^{(0)}, \dots, u^{(j)}\right)$ are the Adomian polynomials with the following explicit representation

$$\begin{aligned} &A_j\left(N; u^{(0)}, \dots, u^{(j)}\right) = \\ &= \sum_{\alpha_1+\dots+\alpha_j=j} N^{(\alpha_1)}\left(u^{(0)}\right)\frac{\left(u^{(1)}\right)^{\alpha_1-\alpha_2}}{(\alpha_1-\alpha_2)!} \times \cdots \\ &\cdots \times \frac{\left(u^{(j-1)}\right)^{\alpha_{j-1}-\alpha_j}}{(\alpha_{j-1}-\alpha_j)!}\frac{\left(u^{(j)}\right)^{\alpha_j}}{(\alpha_j)!}, \end{aligned} \tag{8}$$

where the sequence of indices natural α_i is not increasing, $N^{(i)}(u)$ is the $i-$th (Fréchet) derivative of the operator N.

The solution of (1) can be now represented by (provided that the convergence radius of series (3) is not less then 1)

$$u = u(1) = \sum_{j=0}^{\infty} u^{(j)} \tag{9}$$

and the truncated sum

$$\overset{m}{u} = \sum_{j=0}^{m} u^{(j)} \tag{10}$$

represents an approximation to the exact solution.

The following theorem from [1] gives some sufficient conditions for the convergence of (3) for all $t \in [0,1]$.

Theorem 1. Let H be a Banach space and $F \in H$. If the operator $N(u): H \to H$ is analytic in a ball $\|u - u_0\| < R$ with the center u_0 and if for all $n \geqslant 0$ there holds $\|N^{(n)}(u_0)\| \leqslant n! M\alpha^n$ with some $M > 0$, $\alpha > 0$, then the conditions

1. $4M\alpha \leqslant 1$, for $R = \infty$,
2. $5M\alpha \leqslant 1$, for $R < \infty$.

provide the convergence of (3) for all $t \in [0,1]$ and, therefore, the convergence of (9).

2. Application to parabolic problems

We consider the problem

$$\begin{aligned} &\frac{\partial u(t)}{\partial t} + Au(t) = f(t, u(t)), \quad t \in (0,1], \\ &u(0) = u_0, \end{aligned} \tag{11}$$

where $u(t)$ is an unknown vector valued function with values in a Banach space X, $u_0 \in X$ is a given vector, $f(t,u): (P_+ \times X) \to X$ is a given function (nonlinear operator) and A is a linear densely defined closed operator with the domain $D(A)$ acting in X. The abstract setting (11) covers many applied problems such as nonlinear heat conduction or diffusion in porous media, the flow of electrons and holes in semiconductors, nerve axon equations, chemically reacting systems, equations of the population genetics theory, dynamics of nuclear reactors etc. This fact together with theoretical interest are important reasons to study efficient discrete approximations of problem (11).

A simple example of a partial differential equation covered by the abstract setting (11) is the nonlinear heat equation

$$\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = f(t,x,u) \tag{12}$$

with the initial condition $u(0,x) = u_0(x)$, where the operator A is defined by

$$\begin{aligned} &D(A) = \{v \in H^2(0,1): \ v(0) = 0, v(1) = 0\}, \\ &Av = -\frac{d^2 v}{dx^2} \qquad \text{for all } v \in D(A). \end{aligned} \tag{13}$$

Given a discretization parameter N we are interesting in approximations possessing an exponential convergence rate with respect to $N \to \infty$ which for a given tolerance ε provide algorithms of optimal or low complexity [10]. Exponentially convergent algorithms were proposed recently for various linear problems.

The homogeneous equation

$$\frac{dT(t)}{dt} + AT(t) = 0, \quad T(0) = I, \tag{14}$$

where I is the identity operator and $T(t)$ is an operator valued function defines the semi-group of bounded operators $T(t) = e^{-At}$ generated by A (called also the operator exponential or the solution operator of the homogeneous equation (11)). Given the solution operator, the initial vector u_0 and the right-hand side $f(t)$, the solution of the homogeneous initial value problem (11) can be represented by

$$u(t) = u_h(t) = T(t)u_0 = e^{-At}u_0. \tag{15}$$

Problem (11) is equivalent to the nonlinear Volterra integral equation

$$u(t) = u_h(t) + u_{nl}(t), \tag{16}$$

where $u_h(t) = T(t)u_0$ and

$$T(t) = e^{-At}$$

is the operator exponential (the semi-group) generated by A and the nonlinear term is given by

$$u_{nl}(t) = \int_0^t e^{-A(t-s)} f(s, u(s))ds. \tag{17}$$

The equation (16) is of the type (1) with $u = u(t) = u(t,x)$, $u_0 = u_0(x)$

$$\begin{aligned} N(t,u) &= N(u) = u_{nl} = \\ &= \int_0^t e^{-A(t-s)} f(s,u(s))ds, \\ F &= u_h = F(t,x) = T(t)u_0. \end{aligned} \tag{18}$$

Let A be a densely defined strongly positive (sectorial) operator in a Banach space X with the domain $D(A)$, i.e. its spectrum $\Sigma(A)$ lies in the sector

$$\Sigma = \{z = a_0 + re^{i\theta} : \ r \in [0,\infty), \ |\theta| < \varphi < \frac{\pi}{2}\} \tag{19}$$

and on its boundary Γ_Σ and outside the sector the following estimate for the resolvent holds true

$$\|(zI - A)^{-1}\| \leqslant \frac{M}{1+|z|} \tag{20}$$

with some positive constant M (compare with [18]). The angle φ is called the spectral angle of the operator A. A practically important example of strongly positive operators in $X = L_p(\Omega)$, $0 < p < \infty$ represents a strongly elliptic partial differential operator [5, 9, 6] where the parameters a_0, φ of the sector Σ are defined by its coefficients.

A convenient representation of the operator exponential is the one provided by the improper Dunford-Cauchy integral

$$e^{-At} = \frac{1}{2\pi i}\int_{\Gamma_I} e^{-tz}(zI - A)^{-1}dz \tag{21}$$

where Γ_I is an integration path enveloping the spectrum of A. After parametrizing Γ we get an improper integral of the type

$$\begin{aligned} e^{-At} &= \tfrac{1}{2\pi i}\int_{\Gamma_I} e^{-tz}(zI - A)^{-1}dz = \\ &= \tfrac{1}{2\pi i}\int_{-\infty}^{\infty} \mathscr{F}(t,\xi)d\xi. \end{aligned} \tag{22}$$

The last integral can be discretized by a quadrature rule (desirable exponentially convergent) involving a short sum of resolvents. Such an algorithm inherits a two-level parallelism with respect to both the computation of resolvents and the treatment of different time values.

Two efficient methods for solving linear homogeneous parabolic problems based on the improper Dunford-Cauchy integrals along a path enveloping the spectrum of A were discussed in [6, 10, 12] where the boundary of a sector containing the spectrum of A or a parabola were used as the integration path. The methods from [6] use Sinc-quadratures [23] and possesse the exponential convergence rates for $t > 0$ and a polynomial convergence rates for $t = 0$ depending on the smoothness of the initial vector u_0 from a Hilbert space. An exponential convergence rate for all $t \geqslant 0$ was proved in [7] under assumptions that the initial function u_0 belongs to the domain of $D(A^\sigma)$ for some $\sigma > 1$, where the preliminary computation of $A^\sigma u_0$ is needed. Note that all these algorithms can not be directly applied to inhomogeneous problems due to the inefficiency of computation of the operator exponential at $t = 0$. A hyperbola was used as the integration path which allows one to get the uniform exponential convergence rate with respect to $t \geqslant 0$ without preliminary computation of $A^\sigma u_0$. An exponentially convergent algorithm for the case of an operator family $A(t)$ depending on the parameter t was proposed in [11]. This algorithm uses an exponentially convergent algorithm for the operator exponential generating by a constant operator.

We can also use the representation

$$e^{-At}u_0 = \frac{1}{2\pi i}\int_{\Gamma_I} e^{-zt}\left[(zI-A)^{-1} - \frac{1}{z}I\right]u_0 dz \tag{23}$$

instead of (22), where the integration hyperbola is given by

$$\Gamma_I = \{z(\xi) = a_I\cosh\xi - ib_I\sinh\xi:\ \xi\in(-\infty,\infty)\} \tag{24}$$

(note that the hyperbola

$$\Gamma_0 = \{z(\xi) = a_0\cosh\xi - ib_0\sinh\xi:\ \xi\in(-\infty,\infty), \tag{25}$$

$$b_0 = a_0\tan\varphi\}$$

is called the spectral hyperbola, which pathes through the vertex $(a_0, 0)$ of the spectral angle and possesses asymptotes which are parallel to the rays of the spectral angle Σ).

Parametrizing integral (23) by (24) we get

$$u(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\mathscr{F}(t,\xi)d\xi \tag{26}$$

with

$$\begin{aligned}&\mathscr{F}(t,\xi) = F_A(t,\xi)u_0,\\ &\mathscr{F}_A(t,\xi) = e^{-z(\xi)t}(a_I\sinh\xi - ib_I\cosh\xi)\cdot\\ &\cdot\left[(z(\xi)I-A)^{-1} - \frac{1}{z(\xi)}I\right].\end{aligned} \tag{27}$$

We approximate integral (26) by the following Sinc-quadrature

$$u_M(t) = \frac{h}{2\pi i}\sum_{k=-M}^{M}\mathscr{F}(t, z(kh)) \tag{28}$$

with an appropriate h. The following result from characterizes the error of this approximation.

Theorem 2. Let A be a densely defined strongly positive operator and $u_0 \in D(A^\alpha)$, $\alpha\in(0,1)$, then Sinc-quadrature (28) represents an approximate solution of the homogeneous initial value problem, i.e. $u(t) = e^{-At}u_0$, and possesses a uniform with respect to $t \geqslant 0$ exponential convergence rate with an estimate which is of the order $\mathcal{O}(e^{-c\sqrt{M}})$ uniformly in $t \geqslant 0$ provided that $h = \mathcal{O}(1/\sqrt{M})$ and of the order $\mathcal{O}\left(\max\left\{e^{-\pi dM/(c_1\ln M)},\ e^{-c_1 a_I tM/2 - c_1\alpha\ln M}\right\}\right)$ for each fixed $t \geqslant 0$ provided that $h = (c_1\ln M)/M$.

In accordance with (7) for the computation of Adomian's polynomials one should compute first

$$A_0(N;u_0) = N(t,u_0) = \int_0^t e^{-A(t-\tau)}f(\tau,u_0(\tau))d\tau \tag{29}$$

and then one after another

$$N^{\alpha_1}(t,u_0) = \int_0^t e^{-A(t-\tau)}\frac{\partial^{\alpha_1}f(\tau,u_0(\tau))}{\partial u^{\alpha_1}}d\tau. \tag{30}$$

Using representation (23) of the operator exponential we get, e.g. for $N(t,u_0)$

$$\begin{aligned}N(t,u_0) &= \frac{1}{2\pi i}\times\\ &\times\int_0^t\int_{\Gamma_I} e^{-z(t-s)}\left[(zI-A)^{-1} - \frac{1}{z}I\right]f(s,u_0(s))dzds =\\ &= \frac{1}{2\pi i}\int_{\Gamma_I}\left[(z(\xi)I-A)^{-1} - \frac{1}{z(\xi)}I\right]\times\\ &\times\int_0^t e^{-z(\xi)(t-s)}f(s,u_0(s))dsz'(\xi)d\xi,\\ &z(\xi) = a_I\cosh\xi - ib_I\sinh\xi.\end{aligned} \tag{31}$$

Replacing here the first integral by quadrature (28) we get

$$u_p(t) \approx u_{ap}(t) = \frac{h}{2\pi i}\sum_{k=-M}^{M}z'(kh)\times \tag{32}$$

$$\times\left[(z(kh)I-A)^{-1} - \frac{1}{z(kh)}I\right]f_k(t),\ k = \overline{-M,\ M}$$

with $f_k(t) = \int_0^t e^{-z(kh)(t-s)}f(s,u_0(s))ds$.

In order to construct an exponentially convergent quadrature for these integrals we change the variables by

$$\frac{t}{2} - s = \frac{t}{2}\tanh\xi \tag{33}$$

and get instead of

$$f_k(t) = \int_{-\infty}^{\infty}\mathscr{F}_k(t,\xi)d\xi, \tag{34}$$

where

$$\mathscr{F}_k(t,\xi) = \exp[-z(kh)t(1+\tanh\xi)/2]\times \tag{35}$$

$$\times\frac{t}{2\cosh^2\xi}f(t(1-\tanh\xi)/2, u_0(t(1-\tanh\xi)/2)).$$

The following assertion was proven.

Lemma 1. Let $f(t,u_0(t))$ for $t\in[0,\infty]$ can be analytically extended into the sector $\Sigma_f = \{\rho e^{i\theta_1}:\ \rho\in[0,\infty],\ |\theta_1| < \varphi\}$ and for all complex $w\in\Sigma_f$ we have

$$\|f(w,u_0(w))\| \leqslant ce^{-\delta|\Re w|} \tag{36}$$

with $\delta\in(0,\sqrt{2}a_0]$, then the integrand $\mathscr{F}_k(t,\xi)$ can be analytically extended into the strip D_{d_1}, $0 < d_1 < \varphi/2$ and belongs to the Hardy class $H^1(D_{d_1})$ with respect to ξ, where a_0, φ are the spectral characterizations (19) of A.

Let the assumptions of Lemma 1 hold, then we can use the following quadrature rule to compute the integrals (34).

$$f_k(t) \approx f_{k,M}(t) = h \sum_{p=-M}^{M} \mu_{k,p}(t) f(\omega_p(t)), \qquad (37)$$

where

$$\mu_{k,p}(t) = \frac{t}{2}\exp\{-\frac{t}{2}z(kh)[1+\tanh(ph)]\}/\cosh^2(ph),$$
$$\omega_p(t) = \frac{t}{2}[1-\tanh(ph)],\ h = \mathcal{O}(1/\sqrt{M}),$$
$$z(\xi) = a_I\cosh\xi - ib_I\sinh\xi. \qquad (38)$$

Substituting (37) into (32) we get the following algorithm to compute an approach $A_{0,M}(N;u_0)$ to $A_0(N;u_0)$

$$A_{0,M}(N;u_0) = A_{ap,M}(t) = \\ = \frac{h}{2\pi i}\sum_{k=-M}^{M} z'(kh)\Big[(z(kh)I-A)^{-1}- \\ -\frac{1}{z(kh)}I\Big] h \sum_{p=-M}^{M}\mu_{k,p}(t) f(\omega_p(t), u_0(\omega_p(t))). \qquad (39)$$

The next theorem characterizes the error of this algorithm.

Theorem 3. Let A be a densely defined strongly positive operator with the spectral characterization a_0, φ and $f(t,u_0(t)) \in D(A^\alpha)$, $\alpha > 0$ for $t\in[0,\infty]$ can be analytically extended into the sector $\Sigma_f = \{\rho e^{i\theta_1} : \rho\in[0,\infty],\ |\theta_1|<\varphi\}$ where the estimate

$$\|A^\alpha f(w,u_0(w))\| \leqslant c_\alpha e^{-\delta_\alpha|\Re w|},\ w\in\Sigma_f \qquad (40)$$

with $\delta_\alpha \in (0,\sqrt{2}a_0]$ holds, then algorithm (39) converges with the error estimate

$$\|cE_M(t)\| = \|A_0(N;u_0)(t) - A_{0,M}(t)\| \leqslant ce^{-c_1\sqrt{M}} \qquad (41)$$

uniformly in t with positive constants c, c_1 depending on α, φ, a_0 and independent of M.

In this way one can compute all Adomian's polynomials needed in (10) with an exponential accuracy.

Let the nonlinear operator $f(s,u)$ for each s is analytic as function of u in some disc $\|u-u_0\| = \rho < r$ with the boundary Γ, then it holds

$$|f^{(n)}_{u^n}(s,u_0)| \leqslant \tilde{M}(s,\rho)n!\rho^n \leqslant M(\rho)n!\rho^n, \qquad (42)$$

provided that $\tilde{M}(s,\rho) \leqslant M(\rho)$, where $\tilde{M}(s,\rho) = \max_{u\in\Gamma}\|f(s,u)\|$. Thus, for $M(\rho)$ small enough we are in the situation of Theorem 1, so that the method (10) converges exponentially.

An alternative exponentially convergent method based on the interpolation of the nonlinearity on a Gauss-Lobatto grid was proposed in [13].

One can obtain an operator equation of type (1) applying to PDE (11) the operator $L_t^{-1} = \int_0^t$ analogously to [4, 21] but there are not any theoretical justification of convergence of Adomian's method in this case.

Remark 1. In the recent paper [8] the following modification of ADM was proposed. One looks the summands of (9) in accordance with the recurrence formulas

$$\begin{aligned} &u^{(j+1)} = -\overline{A_j}\left(u^{(0)},u^{(1)},\dots,u^{(j)}\right), \\ &u^{(0)} = F, \quad j=0,1,\dots \end{aligned} \qquad (43)$$

where $\overline{A_j}\left(u^{(0)},u^{(1)},\dots,u^{(j)}\right)$ are the modified Adomian polynomials given by

$$\overline{A_j}\left(u^{(0)},u^{(1)},\dots,u^{(j)}\right) = \qquad (44)$$

$$= N\left(u^{(0)}+\ldots+u^{(j)}\right) - N\left(u^{(0)}+\ldots+u^{(j-1)}\right).$$

In the paper [8] for the problem

$$\frac{d^k y(t)}{dt^k} + \beta(t)f(y(t)) = \varkappa(t), \quad t\in(0,\ T),$$

$$\frac{d^p y(t)}{dt^p} = c_p, \quad p = \overline{0,k-1},$$

with the given c_p, $p=\overline{0,k-1}$, with

$$M = \max_{t\in[0,\ T]}\|\beta(t)|$$

and with the right-hand side $f(y)$ satisfying the Lipschitz condition with a constant L it was shown that the modified Adomian methods converges as a geometrical progression with the quotient α and with the error estimate

$$\left|y(t) - \sum_{j=0}^{m} y_j(t)\right| \leqslant \frac{\alpha^m}{1-\alpha}\|y_1\|_\infty \qquad (45)$$

provided that

$$\alpha = \frac{LMT^k}{k!} < 1. \qquad (46)$$

Numerical experiments have shown that the modified ADM converges faster then the ordinary one. But it was ignored in [8] that the modified ADM in fact coincides with the usual fixed point iteration. Actually, the relations (43), (44) imply

$$\begin{aligned} &u^{m+1} = -N(u^m)+F, \quad m=0,1,\dots \\ &u^0 = F. \end{aligned} \qquad (47)$$

Now the conclusions of [8] about the advantages of the modified ADM become understandable and are well known long ago (see e.g. [22]).

The natural question arises in the case when the assumptions of Theorem 1 are not fulfilled: what can we do in order to arrive the convergence? One of the aims of this paper is to answer this question and to construct an iteration method which converges whereas the fixed point iteration (47) can be divergent. An other aim is to show that the odd and the even iterations provide the two-sided approximations, therefore they can be used e.g. for aposteriori error estimates.

3. Two-sided iteration method

Let $\overline{S}_r(a) = \{x \in X : \ \|x - a\| \leqslant r\}$ be a closed ball in a Banach space X. Then the following assertion about the fixed point iteration for equation (1) holds true (see e.g. [16]).

Theorem 4. Let the operator N satisfies the conditions

1° $\forall u, v \in \overline{S}_r(a)$ it holds

$$\|N(u) - N(v)\| \leqslant q\|u - v\|, \quad q \in (0; 1),$$

2° for $F \in \overline{S}_r(a)$ it holds $\|N(F)\| \leqslant (1 - q)r$.

Then the equation

$$u = -N(u) + F, \quad N(0) = 0 \tag{48}$$

possesses a unique solution $u_\star \in \overline{S}_r(a)$, which can be obtained by the fixed point iteration

$$u_{n+1} = -N(u_n) + F, \quad n = 0, 1, \dots \tag{49}$$

with the error estimate

$$\|u_\star - u_n\| \leqslant \frac{q^n r}{1 - q}.$$

Let us clarify the conditions on the operator N under which the iterations (49) provide the two-sided approximations to $u_\star$.

Let $K \subset X$ be a cone with a partial order $\preceq$, i.e. we write $v \preceq u$ when $u - v \in K$. Further we make the following assumptions.

3° $F \in K$.

4° The operator N is positive in the sense that

$$N(K) \subset K$$

5° There exists the Frechét derivative $N'(v)$ with the property

$$\|N'(v)\| \leqslant q, \ 0 \preceq N'(v)u \quad \forall u, v \in \overline{S}_r(F) \cap K$$

6°

$$0 \preceq u_1 = -N(u_0) + F = -N(F) + F$$

Then the following assertion holds true.

Theorem 5. Let the conditions 2°–6° hold. Then the fixed point iteration (49) converges to the unique solution $u_\star$ of the equation (48) and provide the two-sided approximation, i.e.

$$\begin{array}{l} u_\star \preceq \dots \preceq u_{2k} \preceq \dots \preceq u_2 \preceq u_0 \\ u_1 \preceq u_3 \preceq \dots \preceq u_{2k+1} \preceq \dots \preceq u_\star \end{array} \tag{50}$$

Let us consider the Dirichlet boundary value problem

$$\begin{array}{l} u''(x) - Mu^3(x) = -f(x), \quad x \in (0;\ 1), \\ u(0) = u(1) = 0, \end{array} \tag{51}$$

with $f(x) = \pi^2 \sin \pi x + M(\sin \pi x)^3$ and a given constant $M \geqslant 0$. The exact solution of (51) is

$$u(x) = \sin \pi x. \tag{52}$$

Problem (51) is equivalent to the following nonlinear Fredholm integral equation

$$\begin{array}{l} u(x) = \int\limits_0^1 G(x, \xi)[-Mu^3(\xi) + f(\xi)]d\xi, \\ G(x, \xi) = \left\{ \begin{array}{l} x(1 - \xi), \ x \leqslant \xi \\ \xi(1 - x), \ \xi < x. \end{array} \right. \end{array} \tag{53}$$

This equation is of the form (48) with

$$N(u) = M \int_0^1 G(x, \xi)u^3(\xi)d\xi,$$

$$F = \int_0^1 G(x, \xi)f(\xi)d\xi.$$

For $M = 1$, $r = 1.2$, $q = 0.54$ all assumption of Theorem 5 are fulfilled where

$$X = C[0;\ 1], K = \{v(x) \in C[0;\ 1] : \ v(x) \geqslant 0, x \in [0;\ 1]\}$$

$$\begin{array}{c} \|F\|_\infty = 1.087 \ldots < r, \\ \|N(F) - F\|_\infty = 0.101 \ldots < (1 - q)r, \\ 0 \leqslant N(v)u = 3 \int\limits_0^1 G(x, \xi)v^2(\xi)u(\xi)d\xi \leqslant \\ \leqslant \frac{3r^3}{8} = \frac{3*1.44}{8}r \ = \ qr, \\ u_1 = -N(u_0) + F = -\int\limits_0^1 G(x, \xi)f(\xi)d\xi + f(x) \geqslant 0. \end{array}$$

The numerical results (obtained with Maple) are presented in Table 6. The table exhibits the advantages of the fixed point iteration to the usual Adomian's method.

4. An iteration method for nonlinear problems with the controllable exponential convergence

In this section we give the description of an algorithm with the controllable exponential convergence. In order to avoid technical difficulties we justify this algorithm in the case of the following nonlinear model

n	ε_A	ε_{mA}
0	.4477593957331e-1	.394026825275758e-1
1	.6148944791993e-2	.473705632178769e-2
2	.114506764843e-2	.544572291857201e-3
3	.246398359542e-3	.629246333480784e-4
4	.5768833894e-4	.726644599750120e-5
5	.1427355199e-4	.839176371831571e-6
6	.3671925821e-5	.969127578152334e-7
7	.972342751e-6	.111920351333820e-7

Табл. 1. Error of the usual (ε_A) and the modified (ε_{mA}) Adomian's method for $M = 0.5$.

problem

$$u''(x) - N(u(x))u(x) = -f(x), \quad x \in (0,1), \qquad u(0) = u(1) = 0 \tag{54}$$

with a nonlinear function $N(u) : \mathbf{R}^1 \to \mathbf{R}^1$ satisfying the conditions

$$N(u) \geqslant 0, \quad [uN(u)]' \geqslant 0, \quad N''(u) \geqslant 0, \quad \forall\, u \in \mathbf{R}^1. \tag{55}$$

Using the Green function for the differential operator defined by

$$D(\mathcal{A}) = \{u : u \in W_2^2(0,1) : u(0) = u(1)\},$$

$$\mathcal{A}u = -\frac{d^2u}{dx^2} \;\forall u \in D(\mathcal{A})$$

one can reduce problem (54) to the operator equation of kind (1). In the case when the fixed point iteration (47) is divergent we propose the following method based on the idea of the FD-method [20] which is closed to a homotopy perturbation method. We introduce a grid

$$\overline{\omega} = \left\{x_i \in [0,\ 1],\ i = \overline{1,K}:\ 0 = x_1 < \ldots < x_K = 1\right\}$$

partitioning the interval $[0,\ 1]$ into subintervals $[x_{i-1},\ x_i]$, $i = \overline{1,K}$ of the length $h_i = x_i - x_{i-1}$, $|h| = \max_i h_i$ and imbed problem into the parametric family of problems

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \left\{N(u(x_{i-1},t)) + t\,[N(u(x,t)) - N(u(x_{i-1},t))]\right\}u(x,t) = -f(x), x \in (0,\ 1),$$

$$[u(x)]_{x=x_i} = 0, \quad \left[\frac{du(x)}{dx}\right]_{x=x_i} = 0, \quad i = \overline{1,K-1},$$

$$u(0,t) = u(1,t) = 0, \quad t \in [0,\ 1].$$

It is clear that for $t = 1$ the solution of problem (6) coincides with the solution of problem, i.e.

$$u(x,1) = u(x),$$

For $t = 0$ we obtain the following base problem

$$\begin{aligned} &\frac{d^2u^{(0)}(x)}{dx^2} - N\left(u^{(0)}(x_{i-1})\right)u^{(0)}(x) = -f(x), \\ &\quad x \in (x_{i-1},\ x_i), \quad i = \overline{1,K}, \\ &[u^{(0)}(x)]_{x=x_i} = 0, \quad \left[\frac{du^{(0)}(x)}{dx}\right]_{x=x_i} = 0, \\ &i = \overline{1,K-1}, \\ &u^{(0)}(0) = u^{(0)}(1) = 0, \end{aligned} \tag{56}$$

where $[v(x)]_{x=\xi} = v(\xi+0) - v(\xi-0)$ denotes the jump of the function $v(x)$ at the point $x = \xi$.

The last problem as well as problem (6) are representatives of the class of boundary value problems with piecewise constant argument which are in the focus of attention of many researchers for some time (see e.g. [3] and the literature therein).

We look for the solution of problem (6) in the form

$$u(x,t) = \sum_{j=0}^{\infty} t^j u^{(j)}(x). \tag{57}$$

Substituting (57) into (6) and comparing the coefficients in front of powers of t we obtain the following recurrence sequence of problems for $u^{(j)}(x)$ (with piecewise constant argument):

$$\frac{d^2u^{(j+1)}(x)}{dx^2} - N\left(u^{(0)}(x_{i-1})\right)u^{(j+1)}(x) = \tag{58}$$

$$= N'\left(u^{(0)}(x_{i-1})\right)u^{(j+1)}(x_{i-1})u^{(0)}(x) + F^{(j+1)}(x),$$

$$x \in (x_{i-1}, x_i), \quad i = \overline{1,K},$$

where

$$\begin{gathered} F^{(j+1)}(x) = \\ = \sum_{p=1}^{j} A_{j+1-p}\left(N; u^{(0)}(x_{i-1}), \ldots, u^{(j+1-p)}(x_{i-1})\right) \times \\ \times u^{(p)}(x) + \sum_{p=0}^{j}\left[A_{j-p}\left(N; u^{(0)}(x), \ldots, u^{(j-p)}(x)\right) - \right. \\ \left. - A_{j-p}\left(N; u^{(0)}(x_{i-1}), \ldots u^{(j-p)}(x_{i-1})\right)\right] u^{(p)}(x) + \\ + A_{j+1}\left(N; u^{(0)}(x_{i-1}),\ \ldots, u^{(j)}(x_{i-1}), 0\right)u^{(0)}(x), \\ [u^{(j+1)}(x)]_{x=x_i} = 0, \end{gathered}$$

$$\left[\frac{du^{(j+1)}(x)}{dx}\right]_{x=x_i} = 0, \quad i = \overline{1,K-1},$$

$$u^{(j+1)}(0) = u^{(j+1)}(1) = 0, \quad j = 0, 1, \ldots,$$

$A_j(N; v_0, v_1, \ldots, v_j)$ are Adomian's polynomials for the nonlinear function $N(v)$ given by the explicit formula (8). The solution of problem is then given by

$$u(x) = \sum_{j=0}^{\infty} u^{(j)}(x) \tag{59}$$

(provided that the convergence radius of (57) is not less then 1) and the approximate solution by

$$u(x) \approx \overset{m}{u}(x) = \sum_{j=0}^{m} u^{(j)}(x), \tag{60}$$

where the exponential convergence will be controlled by the parameter $|h|$.

Let us consider the base problem (56). This problem is equivalent to the system of nonlinear equations

$$u^{(0)}(x_i) = \int_0^1 G\left(x_i, \xi, \overrightarrow{N}(u)\right) f(\xi) d\xi, \quad i = \overline{1, K-1}, \tag{61}$$

where $\overrightarrow{N}(u) = \left(N\left(u^{(0)}(x_1)\right), \ldots, N\left(u^{(0)}(x_{K-1})\right)\right)$ and $G\left(x_i, \xi, \overrightarrow{N}\right)$ is the Green function of problem (56) provided that the vector

$$\overrightarrow{u} = \{u^{(0)}(x_1), u^{(0)}(x_2), ..., u^{(0)}(x_{K-1})\}$$

is known.

We introduce the operator

$$B(\overrightarrow{u}) = \left(\int_0^1 G\left(x_i, \xi, \overrightarrow{N}(u)\right) f(\xi) d\xi\right)_{i=1}^{K-1} \tag{62}$$

which is continuous on a closed ball $\overline{S} = \left\{\overrightarrow{u} \in \mathbf{R}^{K-1} : \|\overrightarrow{u}\|_{0,\infty,\hat{\omega}_N} = \max_{1 \leqslant i \leqslant K-1} |u^{(0)}(x_i)| \leqslant r\right\}$ with r defined by $\|f\|_{0,\infty,[0,1]}$ and translate the ball $\overline{S}$ into itself. Therefore, by Brower's fixed point theorem (see e.g. [15]) there exists a fixed point of this operator in $\overline{S}$, i.e. the system of equations (61) is solvable.

Remark 2. The fixed point iteration for equation (61) is equivalent to the solution of sequence of the following problems

$$\begin{gathered}\frac{d^2 u^{(0),n+1}(x)}{dx^2} - N\left(u^{(0),n}(x_{i-1})\right) u^{(0),n+1}(x) = -f(x), \\ x \in (x_{i-1}, \ x_i), \qquad i = \overline{1, K} \\ u^{(0),n+1}(0) = u^{(0),n+1}(1) = 0, \quad n = 0, 1, \ldots,\end{gathered} \tag{63}$$

where $\overrightarrow{u^{(0),0}} = \left(u^{(0),0}(x_i)\right)_{i=\overline{1,K}}$ is an arbitrary vector from the ball $\bar{S}$. For this problem there exists the following exact difference scheme [24]

$$\begin{gathered}\left(a^n(x_i) u^{(0),n+1}(x_i)_{\bar{x}}\right)_x - d^n(x_i) u^{(0),n+1}(x_i) = \\ = -\varphi^n(x_i), \qquad i = \overline{1, K},\end{gathered} \tag{64}$$

$$u^{(0),n+1}(0) = u^{(0),n+1}(1) = 0,$$

with

$$a^n(x_i) = \left[\frac{\sinh(\sqrt{\mu_i^n} h_i)}{h_i \sqrt{\mu_i^n}}\right]^{-1},$$

$$\mu_i^n = N\left(u^{(0),n}(x_{i-1})\right),$$

$$d^n(x_i) = \frac{\sqrt{\mu_i}}{\hbar_i} \tanh \frac{\sqrt{\mu_i} h_i}{2} + \frac{\sqrt{\mu_{i+1}}}{\hbar_i} \tanh \frac{\sqrt{\mu_{i+1}} h_{i+1}}{2},$$

$$\hbar_i = \frac{h_i + h_{i+1}}{2},$$

$$\varphi^n(x_i) = \frac{1}{\hbar_i} \sum_{\alpha=1}^{2} (-1)^\alpha \left[\frac{dW_\alpha^i(x_i)}{dx} + \right.$$

$$\left. +(-1)^\alpha W_\alpha^i(x_i) \sqrt{\mu_{i-1+\alpha}^n} \coth \sqrt{\mu_{i-1+\alpha}^n} h_{i-1+\alpha}\right],$$

where $W_\alpha^i(x), \quad \alpha = 1, 2$ are solutions of the following two Cauchy problems

$$\begin{gathered}\frac{d^2 W_\alpha^j(x)}{dx^2} - N\left(u^{(0),n}(x_i)\right) W_\alpha^j(x) = -f(x), \\ x_{j-2+\alpha} < x < x_{j-1+\alpha}, \\ W_\alpha^j(x_{j+(-1)^\alpha}) = \left.\frac{dW_\alpha^j(x)}{dx}\right|_{x=x_j+(-1)^\alpha} = 0, \\ \alpha = 1, 2.\end{gathered} \tag{65}$$

In order to compute the coefficients of the exact difference scheme for one iteration step one should solve $2(K-1)$ Cauchy problems by an IVP-solver, each on a small interval with the length of the corresponding step-size. Then the difference scheme with a tridiagonal matrix can be solved by the special elimination method (method of chasing, method "progonki") which in our case is stable. Analogously to the multiple shooting method this system of equations can be written down in the form $s = F(s)$ where $s = (s_0^{(1)}, s_1, s_0^{(1)}, ..., s_{K-1}, s_{K-1}^{(1)}, s_K^{(1)})^T$ and $F(s)$ for an arbitrary s can be calculated using an IVP-solver. From the discussion above it follows that the fixed point iteration

$$\overset{m+1}{s} = F(\overset{m}{s}), m = 0, 1, ...$$

converges provided that $\overset{0}{s}$ was chosen within the corresponding ball.

Let us rewrite the equations (58) in the form

$$\begin{aligned}&\frac{d^2 u^{(j+1)}(x)}{dx^2} - q(x) u^{(j+1)}(x) = \\ &= N'\left(u^{(0)}(x_{i-1})\right) [u^{(j+1)}(x_{i-1}) u^{(0)}(x) - \\ &- u^{(j+1)}(x) u^{(0)}(x_{i-1})] + \\ &+ F^{(j+1)}(x), \quad x \in (x_{i-1}, x_i), \quad i = \overline{1, K}, \\ &u^{(j+1)}(0) = u^{(j+1)}(1) = 0, \quad j = 0, 1, \ldots\end{aligned} \tag{66}$$

with

$$\begin{gathered}q(x) = N\left(u^{(0)}(x_{i-1})\right) + N'\left(u^{(0)}(x_{i-1})\right) u^{(0)}(x_{i-1}), \\ x \in [x_{i-1}, x_i), \quad i = \overline{1, K}.\end{gathered} \tag{67}$$

Given $u^{(0)}(x_i), i=1,...,K-1$, let $G(x,\xi,q(\cdot))$ be the Green function corresponding to the operator on the left side of (66) with the Dirichlet boundary conditions. Then problem (66) can be performed to

$$\begin{aligned}&u^{(j+1)}(x)=\sum_{p=1}^{K}\int_{x_{p-1}}^{x_p}G\left(x_i,\xi,q(\cdot)\right)\times\\&\times\int_{x_{p-1}}^{\xi}\frac{du^{(j+1)}(\eta)}{d\eta}d\eta u^{(0)}(\xi)d\xi N'\left(u^{(0)}(x_{k-1})\right)-\\&-\sum_{p=1}^{K}\int_{x_{p-1}}^{x_p}G\left(x_i,\xi,q(\cdot)\right)\int_{x_{p-1}}^{\xi}\frac{du^{(0)}(\eta)}{d\eta}d\eta u^{(j+1)}(\xi)d\xi\times\\&\times N'\left(u^{(j+1)}(x_{p-1})\right)-\int_0^1 G\left(x_i,\xi,q(\cdot)\right)F^{(j+1)}(\xi)d\xi,\\&i=\overline{1,K}.\end{aligned}\tag{68}$$

In order to estimate $u^{(j+1)}(x)$ we need to estimate the Green function $G\left(x,\xi,q(\cdot)\right)$, which can be explicitly represented by the formula

$$G\left(x_i,\xi,q(\cdot)\right)=\frac{1}{v_1(1)}\begin{cases}v_1(x)v_2(\xi), & x\leqslant\xi,\\ v_1(\xi)v_2(x), & \xi\leqslant x.\end{cases}\tag{69}$$

Here $v_\alpha(x),\ \alpha=1,2$ are the so called stencil functions which satisfy the equations

$$\begin{gathered}\frac{d^2}{dx^2}v_\alpha(x)-q(x)v_\alpha(x)=0,\quad 0<x<1,\quad \alpha=1,2,\\ v_1(0)=0,\quad v_1'(0)=1,\quad v_2(1)=0,\quad v_2'(1)=-1\end{gathered}\tag{70}$$

as well as the continuity conditions

$$\begin{gathered}\left[v_\alpha(x)\right]_{x=x_i}=0,\quad \left[v_\alpha'(x)\right]_{x=x_i}=0,\quad \alpha=1,2,\\ i=\overline{1,K-1}.\end{gathered}\tag{71}$$

These functions possess the following properties:

1° $v_1(x)$ is a non decreasing, non negative function on $[0,\ 1]$,
2° $v_2(x)$ is non increasing, non negative function on $[0,\ 1]$,
3° $v_1(1)=v_2(0)$,
4° $v_1'(x)v_2(x)-v_1(x)v_2'(x)\equiv v_1(1)=v_2(0)$,

These properties as well as the maximum principal imply the estimates

$$\begin{gathered}0\leqslant G(x,\xi,q(\cdot))\leqslant G(x,\xi,0),\\ \left|\frac{\partial G(x,\xi,q(\cdot))}{\partial x}\right|\leqslant 1.\end{gathered}\tag{72}$$

Using (72) as well as the assumptions we obtain from (68)

$$\begin{aligned}&\|u^{(j+1)}\|_{1,\infty,[0,1]}\leqslant|h|\|u^{(0)}\|_{1,\infty,[0,1]}\times\\&\times N'\left(\|u^{(0)}\|_{1,\infty,[0,1]}\right)\|u^{(j+1)}\|_{1,\infty,[0,1]}+\\&+\|F^{(j+1)}\|_{1,\infty,[0,1]}.\end{aligned}\tag{73}$$

For $|h|$ small enough this inequality can be transformed to

$$\|u^{(j+1)}\|_{1,\infty,[0,1]}\leqslant c_1\|F^{(j+1)}\|_{1,\infty,[0,1]}\tag{74}$$

with

$$c_1=\left[1-|h|\|u^{(0)}\|_{1,\infty,[0,1]}N'\left(\|u^{(0)}\|_{1,\infty,[0,1]}\right)\right]^{-1}.\tag{75}$$

and the norms

$$\|v\|_{0,\infty,[0,1]}=\max_{x\in[0,1]}|v(x)|,$$

$$\|v\|_{1,\infty,[0,1]}=\max\{\max_{x\in[0,1]}|v(x)|,\max_{x\in[0,1]}|v'(x)|\}.$$

Further we will need the following two auxiliary statements.

Lemma 2. Let $N(u)$ be represented by the power series $N(u)=\sum\limits_{i=1}^{\infty}a_iu^{2i}$, $a_i\geqslant 0$ and $u^{(p)}(x)\in C^1[0,\ 1]$, $p=0,1,\ldots$, then

$$\begin{aligned}&\left\|A_k\left(N(u);u^{(0)}(x),\ldots,u^{(k)}(x)\right)-\right.\\&\left.-A_k\left(N(u);u^{(0)}(x_{i-1}),\ldots,u^{(k)}(x_{i-1})\right)\right\|_\infty\leqslant\\&\leqslant 2h\sum_{i=1}^{\infty}ia_iA_k\left(N(u);\|u^{(0)}\|_{1,\infty,[0,1]},\right.\\&\left.\|u^{(1)}\|_{1,\infty,[0,1]},\ldots,\|u^{(k)}\|_{1,\infty,[0,1]}\right)=\\&=|h|A_k\left(N'(u);\|u^{(0)}\|_{1,\infty,[0,1]},\right.\\&\left.\|u^{(1)}\|_{1,\infty,[0,1]},\ldots,\|u^{(k)}\|_{1,\infty,[0,1]}\right).\end{aligned}$$

Since the Adomian polynomials are linear operators with respect to the first argument (see (8)), i.e.

$$A_k\left(N(u);u^{(0)}(x),\ldots,u^{(k)}(x)\right)=$$

$$=\sum_{i=1}^{\infty}a_iA_k\left(u^{2i};u^{(0)}(x),\ldots,u^{(k)}(x)\right)$$

it is sufficient to consider the case $N(u)=u^{2i}$ only. For this case we have

$$A_k\left(u^{2i};u^{(0)}(x),\ldots,u^{(k)}(x)\right)=$$

$$\begin{aligned}&=\sum_{\alpha_1+\ldots+\alpha_k=k}N^{(\alpha_1)}\left(u^{(0)}(x)\right)\frac{\left[u^{(1)}(x)\right]^{\alpha_1-\alpha_2}}{(\alpha_1-\alpha_2)!}\times\ldots\times\\&\times\frac{\left[u^{(k-1)}(x)\right]^{\alpha_{k-1}-\alpha_k}}{(\alpha_{k-1}-\alpha_k)!}\frac{\left[u^{(k)}(x)\right]^{\alpha_k}}{(\alpha_k)!}=\end{aligned}$$

$$\begin{aligned}&=\sum_{\alpha_1+\ldots+\alpha_k=k}2i(2i-1)\ldots(2i-\alpha_1+1)\times\\&\times\left[u^{(0)}(x)\right]^{2i-\alpha_1}\frac{\left[u^{(1)}(x)\right]^{\alpha_1-\alpha_2}}{(\alpha_1-\alpha_2)!}\times\ldots\times\\&\times\frac{\left[u^{(k-1)}(x)\right]^{\alpha_{k-1}-\alpha_k}}{(\alpha_{k-1}-\alpha_k)!}\frac{\left[u^{(k)}(x)\right]^{\alpha_k}}{(\alpha_k)!},\end{aligned}$$

$$\begin{aligned}&\left\|A_k\left(u^{2i};u^{(0)}(x),\ldots,u^{(k)}(x)\right)-\right.\\&-A_k\left(u^{2i},u^{(0)}(x_{i-1}),\ldots,u^{(k)}(x_{i-1})\right)\big\|\leqslant\\&\leqslant\sum_{\alpha_1+\ldots+\alpha_k=k}2i(2i-1)\ldots(2i-\alpha_1+1)\times\\&\times(2i-\alpha_1+\alpha_1-\alpha_2+\ldots+\alpha_{k-1}-\alpha_k+\alpha_k)\times\\&\times\|u^{(0)}\|^{2i-\alpha_1}_{1,\infty,[0,1]}\frac{\|u^{(1)}\|^{\alpha_1-\alpha_2}_{1,\infty,[0,1]}}{(\alpha_1-\alpha_2)!}\ldots\frac{\|u^{(k-1)}\|^{\alpha_{k-1}-\alpha_k}_{1,\infty,[0,1]}}{(\alpha_{k-1}-\alpha_k)!}\times\\&\times\frac{\|u^{(k)}\|^{\alpha_k}_{1,\infty,[0,1]}}{(\alpha_k)!}|h|=\\&=2i|h|A_k(u^{2i};\|u^{(0)}\|_{1,\infty,[0,1]},\ldots,\|u^{(k)}\|_{1,\infty,[0,1]})=\\&=|h|A_k([u^{2i}]';\|u^{(0)}\|_{1,\infty,[0,1]},\ldots,\|u^{(k)}\|_{1,\infty,[0,1]}).\end{aligned}$$

The lemma is proven.

Lemma 3. Let $N(u)$ be represented by the power series $N(u)=\sum_{j=1}^{\infty}a_ju^{2j}$ then

$$A_{j+1}(N(u);V_0,\ldots,V_j,0)=\frac{1}{(j+1)!}\times\left\{\frac{d^{j+1}}{dz^{j+1}}\left[N(f(z))-(f(z)-V_0)N'(V_0)\right]\right\}_{z=0},\tag{76}$$

with $f(z)=\sum\limits_{j=0}^{\infty}z^jV_j,\quad j=0,1,\ldots.$

Returning to (74) and taking into account we obtain

$$\begin{aligned}&\|u^{(j+1)}\|_{1,\infty}\leqslant c_1\Big\{\sum_{p=1}^{j}A_{j+1-p}(N(u);\|u^{(0)}\|_{1,\infty,[0,1]},\\&\qquad\ldots,\|u^{(j+1-p)}\|_{1,\infty,[0,1]})\|u^{(p)}\|_{1,\infty,[0,1]}+\\&+h\sum_{p=0}^{j}A_{j-p}(N'(u)u;\|u^{(0)}\|_{1,\infty,[0,1]},\ldots,\\&\|u^{(j-p)}\|_{1,\infty,[0,1]})\|u^{(p)}\|_{1,\infty,[0,1]}+\\&+\frac{1}{(j+1)!}\Big[\frac{d^{j+1}}{dz^{j+1}}\Big(N\big(\sum_{s=0}^{\infty}z^s\|u^{(s)}\|_{1,\infty,[0,1]}\big)-\\&-\sum_{s=1}^{\infty}z^s\|u^{(s)}\|_{1,\infty,[0,1]}N'(\|u^{(0)}\|_{1,\infty,[0,1]})\Big)\Big]_{z=0}\Big\}\end{aligned}\tag{77}$$

Introducing in (77) the new variables by

$$|h|^{-j}\,\|u^{(j)}\|_{1,\infty}=v_j,\tag{78}$$

then changing v_j to V_j and the inequality sign to the equality one we arrive at the following system of equations

$$\begin{aligned}&V_{j+1}=c_1\left\{\sum_{p=1}^{j}A_{j+1-p}\left(N(u);V_0,\ldots,V_{j+1-p}\right)V_p+\right.\\&+\sum_{p=0}^{j}A_{j-p}(N'(u)u;V_0,\ldots,V_{j-p})V_p+\\&\left.+\frac{1}{(j+1)!}\frac{d^{j+1}}{dz^{j+1}}\left(N\big(\sum_{s=0}^{\infty}z^sV_s\big)\right)_{z=0}-V_{j+1}N'(V_0)\right\}\\&j=0,1,\ldots,\quad V_0=v_0=\|u^{(0)}\|_{1,\infty,[0,1]}\end{aligned}\tag{79}$$

or

$$\begin{aligned}&V_{j+1}=\tfrac{c_1}{1+c_1N'(V_0)}\left\{\sum_{p=1}^{j}A_{j+1-p}\left(N(u);V_0,\ldots,\right.\right.\\&\left.V_{j+1-p}\right)V_p+\sum_{p=0}^{j}A_{j-p}(N'(u)u;V_0,\ldots,V_{j-p})V_p+\\&\left.+\tfrac{1}{(j+1)!}\tfrac{d^{j+1}}{dz^{j+1}}\left(N\big(\sum_{s=0}^{\infty}z^sV_s\big)\right)_{z=0}\right\}.\end{aligned}\tag{80}$$

The solution of this system is a majorant for the solution of (77), i.e. $v_j\leqslant V_j$, $j=0,1,\ldots$. Using the method of generating functions we obtain from (80)

$$\begin{aligned}&f(z)-V_0=\tfrac{c_1}{1+c_1N'(V_0)}\left\{[f(z)-V_0]\left[N(f(z))-\right.\right.\\&\left.\left.-N(V_0)\right]+zf^2(z)N'(f(z))+N(f(z))-N(V_0)\right\}.\end{aligned}\tag{81}$$

From this equation we can express z as a function of f

$$\begin{aligned}&z=\tfrac{1}{f^2N'(f)}\left\{\left(\tfrac{1}{\tilde{C}}-N(f)+N(V_0)\right)(f-V_0)-\right.\\&\left.-N(f)+N(V_0)\right\},\\&V_0\leqslant f,\quad \tilde{C}=\tfrac{c_1}{1+c_1N'(V_0)},\end{aligned}\tag{82}$$

and then find f_m, for which z arrives its maximum $z_m=R$. The condition

$$|h|V_0\left[N'(V_0)\right]^2<1\tag{83}$$

guarantees the existence of f_m because under assumption (83) we have

$$z(V_0)=0,\quad \lim_{f\to\infty}z(f)=0$$

$$\begin{aligned}&\frac{d}{df}\left[z(f)f^2N'(f)\right]\big|_{f=V_0}=\frac{1}{\tilde{c}}-N'(V_0)=\frac{1}{c_1}=\\&=1-|h|\,\|u^{(0)}\|_{1,\infty,[0,1]}N'(\|u^{(0)}\|_{1,\infty,[0,1]})>0.\end{aligned}$$

The value z_m defines the convergence radius of series (81), i.e.

$$R^jV_j=C\frac{1}{(j+1)^{1+\varepsilon}},\tag{84}$$

with an arbitrarily small positive ε. Returning to the old notations we have

$$\|u^{(j)}\|_{1,\infty,[0,1]}\leqslant\frac{C}{(j+1)^{1+\varepsilon}}\left(\frac{h}{R}\right)^j,\quad j=0,1,\ldots,\tag{85}$$

which leads to the following sufficient convergence condition for the series $f(z)=\sum\limits_{j=0}^{\infty}z^jV_j$:

$$\frac{h}{R}\leqslant 1.\tag{86}$$

Thus, we have proved the following assertion.

Theorem 6. Under the assumptions of Lemma 2 the method (60) for problem (81) converges superexponentially (converges) with the error estimate

$$\|u-\overset{m}{u}\|_{1,\infty,[0,1]}\leqslant\tfrac{C}{(1+m)^{1+\varepsilon}}\tfrac{(h/R)^{m+1}}{1-h/R},\quad or\ \Big(C\sum_{j=m+1}^{\infty}\tfrac{1}{(j+1)^{1+\varepsilon}}\Big)\tag{87}$$

provided that

$$h<R,\ or\ (h=R).\tag{88}$$

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