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A one criterion for the improved regular growth of entire functions with zeros on a finite system of rays

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We establish a criterion for the improved regular growth of entire functions of positive order in terms of Fourier-Stieltjes coefficients of the sequence of their zeros that they are located on a finite system of rays.

In the papers [1, 2], the notion of an entire function of improved regular growth was introduced, and a criteria for this regularity were established in terms of the distribution of zeros that they are located on a finite number of rays. In [3], this notion was generalized to subharmonic functions. Criterion for the improved regular growth of entire functions of positive order with zeros on a finite system of rays in terms of their Fourier coefficients was established in [4].

An entire function f is called a function of *improved regular growth* (see [1, 2]) if for some $\rho \in (0, +\infty)$ and $\rho_1 \in (0, \rho)$, and a 2π-periodic ρ-trigonometrically convex function $h(\varphi) \not\equiv -\infty$ there exists the set $U \subset \mathbb{C}$ contained in the union of disks with finite sum of radii and such that

$$\log |f(z)| = |z|^\rho h(\varphi) + o(|z|^{\rho_1}), \quad U \not\ni z = re^{i\varphi} \to \infty.$$

If f is an entire function of improved regular growth, then it has the order ρ and indicator h [1].

Let f be an entire function with $f(0) = 1$, let $(\lambda_n)_{n\in\mathbb{N}}$ be the sequence of its zeros, let p be the least nonnegative integer number for which $\sum_{n=1}^{\infty} |\lambda_n|^{-p-1} < +\infty$, let $n(r, \psi; f) := \sum_{|\lambda_n|\leqslant r, \arg\lambda_n=\psi} 1$, let [5, p. 104] $n_k(r, f) := \sum_{|\lambda_n|\leqslant r} e^{-ik\arg\lambda_n}$, $k \in \mathbb{Z}$, and let Q_ρ be the coefficient of z^ρ in the exponential factor in the Hadamard-Borel representation [6, p. 38] of an entire function f of order $\rho \in (0, +\infty)$.

Theorem A [1]. *An entire function f of noninteger order $\rho \in (0, +\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1, \dots, m\}$, $0 \leqslant \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$, is a function of improved regular growth if and only if, for a certain $\rho_2 \in (0, \rho)$ and each $j \in \{1, \dots, m\}$, one has*

$$n(t, \psi_j; f) = \Delta_j t^\rho + o(t^{\rho_2}), \quad \Delta_j \in [0, +\infty), \qquad (1)$$

as $t \to +\infty$. In this case,

$$h(\varphi) = \sum_{j=1}^{m} h_j(\varphi),$$

where $h_j(\varphi)$ is the 2π-periodic function defined on the interval $[\psi_j, \psi_j + 2\pi)$ by the equality $h_j(\varphi) = \frac{\pi\Delta_j}{\sin\pi\rho}\cos\rho(\varphi - \psi_j - \pi)$.

Theorem B [2]. *An entire function f of order $\rho \in \mathbb{N}$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1, \dots, m\}$, $0 \leqslant \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$, is a function of improved regular growth if and only if equality (1) holds for a certain $\rho_2 \in (0, \rho)$ and each $j \in \{1, \dots, m\}$ and, for certain $\rho_3 \in (0, \rho)$ and $\delta_f \in \mathbb{C}$, one has*

$$\sum_{|\lambda_n|\leqslant r} \lambda_n^{-\rho} = \delta_f + o(r^{\rho_3 - \rho}), \quad r \to +\infty. \qquad (2)$$

In this case,

$$h(\varphi) = \begin{cases} \tau_f \cos(\rho\varphi + \theta_f) + \sum_{j=1}^{m} h_j(\varphi), & \rho = p, \\ Q_\rho \cos\rho\varphi, & \rho = p + 1, \end{cases}$$

where $\tau_f = |\delta_f/\rho + Q_\rho|$, $\theta_f = \arg(\delta_f/\rho + Q_\rho)$ and $h_j(\varphi)$ is the 2π-periodic function defined on the interval $[\psi_j, \psi_j + 2\pi)$ by the equality $h_j(\varphi) = \Delta_j(\pi - \varphi + \psi_j)\sin\rho(\varphi - \psi_j) - \frac{\Delta_j}{\rho}\cos\rho(\varphi - \psi_j)$.

The aim of this paper is to prove the following theorems, which improve Theorems A and B.

Theorem 1. *An entire function f of noninteger order $\rho \in (0, +\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1, \dots, m\}$, $0 \leqslant \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$, is a function of improved regular growth if and only if, for certain $\rho_2 \in (0, \rho)$ and $k_0 \in \mathbb{Z}$, and each $k \in \{k_0, k_0 + 1, \dots, k_0 + m - 1\}$, one has*

$$n_k(r, f) = \Delta_k r^\rho + o(r^{\rho_2}), \quad r \to +\infty, \quad \Delta_k \in \mathbb{C}. \qquad (3)$$

Proof. *Necessity.* If f is an entire function of improved regular growth with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1, \dots, m\}$, $0 \leqslant \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$, then by Theorem A the relations (1) hold for certain $\rho_2 \in (0, \rho)$ and each $j \in \{1, \dots, m\}$. Since $n_k(r, f) = \sum_{j=1}^{m} e^{-ik\psi_j} n(r, \psi_j; f)$, then for each $k \in \mathbb{Z}$ the relations (3) hold with

$$\Delta_k = \sum_{j=1}^{m} \Delta_j e^{-ik\psi_j}. \qquad (4)$$

Sufficiency. Let the relations (3) hold with (4). Without loss of generality we can assume that $k_0 = 0$. Then for $k \in \{0, 1, \dots, m - 1\}$, we have (see [4, 5, 7])

$$\begin{cases} n_0(r,f) = n(r,\psi_1;f) + n(r,\psi_2;f) + \ldots + n(r,\psi_m;f), \\ n_1(r,f) = e^{-i\psi_1} n(r,\psi_1;f) + e^{-i\psi_2} n(r,\psi_2;f) + \ldots + e^{-i\psi_m} n(r,\psi_m;f), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ n_{m-1}(r,f) = e^{-i(m-1)\psi_1} n(r,\psi_1;f) + e^{-i(m-1)\psi_2} n(r,\psi_2;f) + \ldots + e^{-i(m-1)\psi_m} n(r,\psi_m;f). \end{cases}$$

This is a system of linear equations with respect to the unknowns $n(r,\psi_j;f)$, $j\in\{1,\ldots,m\}$. Its determinant is the Vandermonde determinant, which distinct from zero. Therefore, the functions $n(r,\psi_j;f)$, $j\in\{1,\ldots,m\}$, can be represented as a linear combinations of functions $n_k(r,f)$, $k\in\{0,1,\ldots,m-1\}$. Solving the considered system by Cramer's rule and using (3), we get (1). Then, by Theorem A, we obtain the required proposition. Theorem 1 is proved.

Note that the similar result for entire functions of strongly regular growth of zero order was obtained in [7].

Theorem 2. *An entire function f of order $\rho\in\mathbb{N}$ with zeros on a finite system of rays $\{z:\arg z=\psi_j\}$, $j\in\{1,\ldots,m\}$, $0\leqslant\psi_1<\psi_2<\ldots<\psi_m<2\pi$, is a function of improved regular growth if and only if the relations (3) hold for certain $\rho_2\in(0,\rho)$ and $k_0\in\mathbb{Z}$, and each $k\in\{k_0,k_0+1,\ldots,k_0+m-1\}$, and, in addition, the equality (2) holds for certain $\rho_3\in(0,\rho)$ and $\delta_f\in\mathbb{C}$.*

The proof of Theorem 2 is based on Theorems 1 and B. Theorem 1 is unimprovable in the sense of the following theorem.

Theorem 3. *For every $m\in\mathbb{N}\backslash\{1\}$ there exists an entire function f of noninteger order $\rho\in(0,+\infty)$ with zeros on a finite system of rays $\{z:\arg z=\psi_j\}$, $\psi_j:=\frac{2\pi(j-1)}{m}$, $j\in\{1,\ldots,m\}$, such that*

$$n_0(r,f) = mr^\rho - m\frac{r^\rho}{\log r} + o\left(\frac{r^\rho}{\log r}\right), \quad r\to+\infty,$$

for any $\rho_2\in(0,\rho)$ and each $k\in\{1,\ldots,m-1\}$ hold (3) and f is not a function of improved regular growth.

Proof. Let $\rho\in(0,+\infty)$ is noninteger, $\mu_n=\big(n+\frac{n}{\log n}\big)^{1/\rho}$, $\{\lambda_n : n\in\mathbb{N}\backslash\{1\}\} := \bigcup_{j=1}^m \{\mu_n e^{i\frac{2\pi(j-1)}{m}} : n\in\mathbb{N}\backslash\{1\}\}$, and

$$f(z)=\prod_{n=1}^{\infty}\left(1-\frac{z}{\lambda_n}\right)\exp\left(\sum_{\nu=1}^{p}\frac{1}{\nu}\left(\frac{z}{\lambda_n}\right)^{\nu}\right),\ p=[\rho].$$

Then for each $j\in\{1,\ldots,m\}$ [4]

$$n\left(t,\frac{2\pi(j-1)}{m};f\right)=t^\rho-\frac{t^\rho}{\rho\log t}+o\left(\frac{t^\rho}{\log t}\right),$$

as $t\to+\infty$. Hence, the equality (1) does not hold for any $\rho_2\in(0,\rho)$ and, according to Theorem A, an entire function f is not a function of improved regular growth. Besides, $n_0(r,f)=\sum_{j=1}^m n\big(r,\frac{2\pi(j-1)}{m};f\big) = mr^\rho - m\frac{r^\rho}{\log r}+o\big(\frac{r^\rho}{\log r}\big)$ as $r\to+\infty$. Therefore, for $k=0$ the relation (3) does not hold. Since

$$\sum_{j=1}^m e^{-ik\frac{2\pi(j-1)}{m}}=\frac{1-e^{-2\pi ki}}{1-e^{-i\frac{2\pi k}{m}}}=0,\ k\in\{1,\ldots,m-1\},$$

then

$$n_k(r,f)=\sum_{\mu_n\leqslant r}\sum_{j=1}^m e^{-ik\frac{2\pi(j-1)}{m}}=0,$$

for each $k\in\{1,\ldots,m-1\}$ and all $r>0$. Thus, the relations (3) hold for any $\rho_2\in(0,\rho)$ and each $k\in\{1,\ldots,m-1\}$. Theorem 3 is proved.

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