UDC 512.4

Module Filters

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Module filters are studied. Description of filters in semisimple modules are given.

Key words and phrases: ring, module, preradical.

All rings are considered to be associative with unit $1 \neq 0$ and all modules are left unitary.

Let R be a ring. The category of left R-modules will be denoted by $R-\mathrm{Mod}$. We shall write $N \leq M$ if N is a submodule of M.

The set of all R-endomorphisms of M will be denoted by $\mathrm{End}(M)$.

Let $\mathrm{soc}(M)$ denote the socle of M and $J(M)$ denote the Jacobson radical of M.

Let $N \leq M$ and $f \in \mathrm{End}(M)$.

Put

$$(N:f)_M = \{x \in M \mid f(x) \in N\},$$

$$\mathrm{End}(M)_N = \{f \in \mathrm{End}(M) \mid f(M) \subseteq N\}.$$

Let E be some non-empty set of submodules of a left R-module M.

Consider the following conditions:

(1) $L \in E, L \leq N \leq M \Rightarrow N \in E$;

(2) $L \in E, f \in \mathrm{End}(M) \Rightarrow (L:f)_M \in E$;

(3) $N, L \in E \Rightarrow N \bigcap L \in E$;

(4) $N \in E, N \in \mathrm{Gen}(M), L \leq N \leq M \wedge$

$\forall g \in \mathrm{End}(M)_N : (L:g)_M \in E \Rightarrow L \in E$;

(5) $N, L \in E, N \in \mathrm{Gen}(M) : N \bigcap L \in E$.

Definition. A non-empty set E of submodules of a left R-module M satisfying (1), (2), (3) is called a preradical filter of M.

Definition. A non-empty set E of submodules of a left R-module M satisfying (1), (2), (4) is called a radical filter of M.

Definition. A preradical (radical) filter E of a left R-module M is said to be trivial if either $E = \{L \mid L \leq M\}$ or $E = \{M\}$.

Let M be a semisimple left R-module with a unique homogeneous component and let $M = \bigoplus_{i \in I} M_i$, where M_i is simple for each $i \in I$.

If $N = \bigoplus_{i \in J} N_i$, where N_i is simple for each $i \in J$ and $M \cong N$, then $\mathrm{Card}(I) = \mathrm{Card}(J)$.

Put

$$\mathrm{Card}_s(M) := \mathrm{Card}(I).$$

Proposition 1. Let M be a semisimple R-module with a unique homogeneous component. If $\mathrm{Card}_s(M)$ is infinite, then every non-trivial radical [preradical] filter of M is of the form

$$E_p(M)$$

for some infinite cardinal number $p \leq \mathrm{Card}_s(M)$.

Proof.

Let M be a semisimple R-module with a unique homogeneous component, $\mathrm{Card}_s(M)=\infty$, and E a non-trivial radical [preradical] filter of M. Put

$$q := \mathrm{Card}_s(M).$$

It is obvious that for each $L\in E$ there exists $H\le M$ such that $M=L\oplus H$. Hence $\mathrm{Card}_s H\le q$.

We claim that $\mathrm{Card}_s H\ne q$. Indeed, suppose, contrary to our claim, that $\mathrm{Card}_s H=q$. Since M is a semisimple R-module with a unique homogeneous component, for some set I we have that $M=\underset{i\in I}{\oplus}M_i$, where M_i is simple for each $i\in I$ and for every $i,j\in I$ there exists an isomorphism $f_{ij}:M_i\to M_j$. Hence $\mathrm{Card}I=\mathrm{Card}_s(M)=q$. Taking into account that $\mathrm{Card}_s(M)$ is infinite,

by (2.1) [p. 417, 2],

$$q+q=q.$$

Consider a set X such that $\mathrm{Card}X=q$ and $X\cap I=\varnothing$. Since $q+q=q$, there exists a bijection $w:X\cup I\to I$. Put

$$Y:=w(X), Z:=w(I).$$

Therefore, $I=Y\cup Z, Y\cap Z=\varnothing, q=\mathrm{Card}I=\mathrm{Card}Y=\mathrm{Card}Z$. Now we obtain $M=A\oplus B$, where $A=\underset{i\in Y}{\oplus}M_i$, $B=\underset{i\in Z}{\oplus}M_i$. Since $H\le M$, there exists an isomorphism $u:H\to\underset{i\in T}{\oplus}M_i$ for some $T\subseteq I$ (see Proposition 9.4 [1]). It is clear that $\mathrm{Card}_s H=\mathrm{Card}T=q$. Whence $q=\mathrm{Card}Y=\mathrm{Card}Z=\mathrm{Card}T$. Let $g:Y\to T, c:Z\to T$ be bijections. Consider the following maps:

$$G:A\to H, C:B\to H,$$

where

$$G(\sum_{i\in Y}m_i)=u^{-1}(\sum_{i\in Y}f_{i,g(i)}(m_i)),$$
$$(m_i\in M_i(i\in I), \mathrm{Card}\{i\in Y|m_i\ne 0\}<\infty),$$

$$C(\sum_{i\in Z}m_i)=u^{-1}(\sum_{i\in Z}f_{i,c(i)}(m_i)),$$
$$(m_i\in M_i(i\in I), \mathrm{Card}\{i\in Z|m_i\ne 0\}<\infty).$$

It is easily seen that these maps are isomorphisms. Let $n,r:M\to M$ are maps such that $n(a+b)=G(a),(a\in A,b\in B)$ and $r(a+b)=C(b),(a\in A,b\in B)$. It is clear that $n,r:M\to M$ are endomorphisms. Since $L\cap H=0$ and G,C are isomorphisms, $(L:n)_M=B$ and $(L:r)_M=A$. As $L\in E$, by (2), we get $B\in E$ and $A\in E$. By (3) or (5), $0=A\cap B\in E$. Consequently, E is trivial. This contradicts our assumption. Hence $\mathrm{Card}_s H<q$. The natural isomorphism $H\cong M/L$ implies that $\mathrm{Card}_s(M/L)<q$. Now we consider the set Ω of all cardinal numbers v such that

$$v\le q \ \ \& \ \ \forall L\in E:\mathrm{Card}_s(M/L)<v.$$

$\Omega\ne\varnothing$, because $q\in\Omega$. By II.15.IV [3], there exists the least element p belonging to Ω. Thus $\forall L\in E:\mathrm{Card}_s(M/L)<p$. It means that $E\subseteq E_p(M)$.

Let $L\in E_p(M)$. Whence $\mathrm{Card}_s(M/L)<p$. We claim that there exists $D\in E$ such that $\mathrm{Card}_s(M/L)\le\mathrm{Card}_s(M/D)$. Conversely, suppose that

$$\forall D\in E:\mathrm{Card}_s(M/D)<\mathrm{Card}_s(M/L).$$

But $\mathrm{Card}_s(M/L)<p\le q$. Hence $\mathrm{Card}_s(M/L)\in\Omega$. Since p is the least element belonging to Ω, $p\le\mathrm{Card}_s(M/L)$, contrary to $\mathrm{Card}_s(M/L)<p$.

Now we have that there exists $D\in E$ such that $\mathrm{Card}_s(M/L)\le\mathrm{Card}_s(M/D)$. It is easily seen that for L,D there exist $H,K\le M$ such that $M=L\oplus H, M=D\oplus K$.

Since $M/L\cong H$, $M/D\cong K$, $\mathrm{Card}_s(H)\le\mathrm{Card}_s(K)$. Since $H\le M$ and $K\le M$, there exist isomorphisms $u:H\to\underset{i\in T}{\oplus}M_i$ for some $T\subseteq I$ and $w:K\to\underset{i\in S}{\oplus}M_i$ for some $S\subseteq I$. Therefore $\mathrm{Card}T\le\mathrm{Card}S$. From this we have that there exists an injective map $\gamma:T\to S$.

Consider the following map:

$$\psi : \underset{i\in T}{\oplus} M_i \to \underset{i\in S}{\oplus} M_i ,$$

where

$$\psi(\sum_{i\in T} m_i) = \sum_{i\in Y} f_{i,\gamma(i)}(m_i) ,$$
$$(m_i \in M_i (i \in I), \mathrm{Card}\{i \in T | m_i \neq 0\} < \infty) .$$

It is obvious that ψ is a monomorphism.

Now consider the following map:

$$\eta : M \to M ,$$

where

$$\eta(l+h) = w^{-1}\psi u(h) ,\ (l \in L, h \in H) .$$

It is clear that $\eta \in \mathrm{End}(M)$. Since $D \cap K = 0$ and $\mathrm{im}\eta \subseteq K$, for every $l \in K, h \in H$: $\eta(l+h) \in D \Leftrightarrow w^{-1}\psi u(h) \in D \Leftrightarrow w^{-1}\psi u(h) = 0$. Since u, w are isomorphisms and ψ is monomorphism, for every $h \in H$: $w^{-1}\psi u(h) = 0 \Leftrightarrow h = 0$. From the above it follows that $(D:\eta)_M = L$. Since E is a radical [preradical] filter of M and $D \in E$, $(D:\eta)_M = L$ shows that $L \in E$, by (2). It means that $E_p(M) \subseteq E$. But $E \subseteq E_p(M)$. Hence $E = E_p(M)$.

Proposition 2. If M is a left R-module such that $M = M_1 \oplus M_2 \oplus ... \oplus M_n$, where $M_i = \mathrm{Tr}_M(M_i)$ for each $i \in \{1,2,...,n\}$ and $\forall S : S \le M \Rightarrow S \in \mathrm{Gen}(M)$, then every radical[preradical] filter E of M is of the form

$$E = \{J_1 + J_2 + ... + J_n | J_i \in E_i (i \in \{1,2,...,n\})\} ,$$

where E_i is a radical [preradical] filter of M_i for each $i \in \{1,2,...,n\}$.

Proof. Let E be a radical[preradical] filter of M and $M = M_1 \oplus M_2 \oplus ... \oplus M_n$, where $M_i = \mathrm{Tr}_M(M_i)$ for each $i \in \{1,2,...,n\}$. Put

$$E_i := \{f_i(K) | K \in E\}$$

for each $i \in \{1,2,...,n\}$, where

$f_i : M \to M, f_i(m_1 + m_2 + ... + m_n) = m_i$, $(m_1 \in M_1, m_2 \in M_2, ..., m_n \in M_n)$ for each $i \in \{1,2,...,n\}$.

(1) Let $L \in E_i, L \le N \le M_i$. Hence there exists $P \in E$ such that $L = f_i(P)$. Since $L \le N$, $P \le f_i^{-1}(N)$. By (1), $f_i^{-1}(N) \in E$, because $P \in E$. Therefore $N = f_i(f_i^{-1}(N)) \in E_i$.

(2) Let $L \in E_i, f \in \mathrm{End}(M_i)$. Hence there exists $P \in E$ such that $L = f_i(P)$. Consider

$F : M \to M$,

where $F : m_1 + m_2 + ... + m_i + ... + m_n \mapsto f(m_i)$, $(m_1 \in M_1, ..., m_n \in M_n)$. Thus $F \in \mathrm{End}(M)$.

We claim that $f_i((P:F)_M) \le (L:f)_{M_i}$. Indeed, let $x_i \in f_i((P:F)_M)$. We have that $x_i \in M_i$. Thus there exists $x \in (P:F)_M$ such that $f_i(x) = x_i$. Hence $f(x_i) = F(x) \in P$. It is clear that $f(x_i) \in M_i$. Therefore $f(x_i) = f_i(f(x_i)) \in f_i(P) = L$. Whence $x_i \in (L:f)_{M_i}$. We obtain $f_i((P:F)_M) \le (L:f)_{M_i}$.

Since $P \in E$ and $F \in \mathrm{End}(M)$, $(P:F)_M \in E$, by (2). $(P:F)_M \in E$ implies $f_i((P:F)_M) \in E_i$. Since $f_i((P:F)_M) \le (L:f)_{M_i}$, (1) implies $(L:f)_{M_i} \in E_i$.

(3) Let $L, N \in E_i$. Hence there exist $P, T \in E$ such that $L = f_i(P)$ and $N = f_i(T)$. By (3) (for preradical filter E), $P \cap T \in E$. Therefore $f_i(P \cap T) \in E_i$.

Since $f_i(P \cap T) \subseteq f_i(P) \cap f_i(T) = L \cap N$ and $f_i(P \cap T) \in E_i$, we obtain $L \cap N \in E_i$, by (1).

(4)Let
$N \in E_i, N \in \mathrm{Gen}(M_i), L \le N \le M_i \wedge \forall g \in \mathrm{End}(M_i)_N : (L:g)_{M_i} \in E_i$.

Hence $N = f_i(T)$ for some $T \in E$. Since $T \subseteq f_i^{-1}(N)$, $f_i^{-1}(N) \in E$, by (1). And $f_i^{-1}(N) \in \mathrm{Gen}(M)$. $L \le N$ implies $f_i^{-1}(L) \le f_i^{-1}(N)$. Let G be an arbitrary element of $\mathrm{End}(M)_{f_i^{-1}(N)}$. By Proposition 8.16 [1], $M_s = \mathrm{Tr}_M(M_s)$ is fully invariant submodule of M for each $s \in \{1,2,\dots,n\}$. Hence $G(M_s) \subseteq M_s$ for each $s \in \{1,2,\dots,n\}$.

Consider

$$g : M_i \to M_i, m \mapsto G(m), (m \in M_i).$$

Since $\forall g \in End(M_i)_N : (L:g)_{M_i} \in E_i$, there exists $Y_g \in E_i$ such that $g(Y_g) \le L$. Since $G(M_s) \subseteq M_s$ for each $s \in \{1,2,\dots,n\}$,

$$G(f_i^{-1}(Y_g)) = G(M_1 \oplus \dots \oplus M_{i-1} \oplus Y_g \oplus \oplus M_{i+1} \oplus \dots \oplus M_n) \subseteq M_1 \oplus \dots \oplus M_{i-1} \oplus$$

$$\oplus G(Y_g) \oplus M_{i+1} \oplus \dots \oplus M_n =$$

$$= M_1 \oplus \dots \oplus M_{i-1} \oplus g(Y_g) \oplus M_{i+1} \oplus \dots \oplus M_n \subseteq$$

$$\subseteq M_1 \oplus \dots \oplus M_{i-1} \oplus L \oplus M_{i+1} \oplus \dots \oplus M_n = f_i^{-1}(L).$$

Hence $f_i^{-1}(Y_g) \subseteq (f_i^{-1}(L):G)_M$. Since $Y_g \in E_i$, there exists $P \in E$ such that $Y_g = f_i(P)$. Thus $P \subseteq f_i^{-1}(Y_g)$. Hence $P \subseteq (f_i^{-1}(L):G)_M \,\&\, P \in E$. By (1), $(f_i^{-1}(L):G)_M \in E$. Since $f_i^{-1}(N) \in E$, $f_i^{-1}(N) \in Gen(M)$, $f_i^{-1}(L) \le f_i^{-1}(N) \le M$ and $\forall G \in End(M)_{f_i^{-1}(N)} : (f_i^{-1}(L):G)_M \in E$, obtain $f_i^{-1}(L) \in E$. Therefore $L = f_i(f_i^{-1}(L)) \in E_i$.

Let $J \in E$. Put $J_i := f_i(J), (i \in \{1,2,\dots,n\})$. By Proposition 8.20 [1], $Tr_J(M) = Tr_J(M_1 \oplus M_2 \oplus \dots \oplus M_n) = \sum_{i=1}^{n} Tr_J(M_i)$. Since $J \le M$, $Tr_J(M_i) \le Tr_M(M_i) = M_i$ for any $i \in \{1,2,\dots,n\}$, by Proposition 8.16 [1]. Hence $Tr_J(M) = \bigoplus_{i=1}^{n} Tr_J(M_i)$, because $M = M_1 \oplus M_2 \oplus \dots \oplus M_n$. Since $J \in Gen(M)$, $Tr_J(M) = J$, by Proposition 8.12.1. Whence $J = \bigoplus_{i=1}^{n} Tr_J(M_i) \,\&\, \forall i \in \{1,2,\dots,n\} : Tr_J(M_i) \le M_i$. Therefore $Tr_J(M_i) = J_i$ for any $i \in \{1,2,\dots,n\}$. Thus $J = J_1 + J_2 + \dots + J_n$, where $J_1 \in E_1, J_2 \in E_2, \dots, J_n \in E_n$.

Let $P_i \in E_i$ for each $i \in \{1,2,\dots,n\}$. Hence there exists $H_i \in E$ such that $P_i = f_i(H_i)$. Thus $H_i \subseteq f_i^{-1}(P_i)$. By (1), $f_i^{-1}(P_i) \in E$. $f_i^{-1}(P_i) \in Gen(M)$ for any $i \in \{1,2,\dots,n\}$. By (3) or (5), $f_1^{-1}(P_1) \cap f_2^{-1}(P_2) \cap \dots \cap f_n^{-1}(P_n) \in E$. Since $f_i^{-1}(P_i) = M_1 + \dots + M_{i-1} + P_i + M_{i+1} + \dots + M_n$ for any $i \in \{1,2,\dots,n\}$, $P_1 + P_2 + \dots + P_n \in E$.

Proposition 3. Let M is a left R-module with $J(M) \ne M$. Then every preradical filter of M is trivial if and only if M is a finitely generated semisimple module M with exactly one homogeneous component.

Proof. $(\Rightarrow)$ Assume that every preradical filter of M is trivial. Let Ss be the class of all semisimple modules of M. Consider

$$F := \{L \le M \mid M/L \in Ss\}.$$

Since $J(M) \ne M$, $F \ne \{M\}$.

(1) Let $L \le K, L \in F$. Then there exists an exact sequence $M/L \to M/K \to 0$. Hence $K \in F$.

(2) Let $L \in F, f \in End(M)$. Since there exists an exact sequence $0 \to M/(L:f)_M \to M/L$, $(L:f)_M \in F$.

(3) Let $L, N \in F$. Since there exists an exact sequence $0 \to M/(L \cap N) \to M/L \times M/N$, $L \cap N \in F$.

Therefore F is a preradical filter.

Since F is a preradical filter and $F \ne \{M\}$, $0 \in F$. Hence M is semisimple.

We shall show that all minimal submodules of M are isomorphic. Suppose that L, N are non-isomorphic minimal submodules of M. Hence $Tr_M(L), Tr_M(N)$ are fully invariant submodules of M. Since L, N are non-isomorphic, $Tr_M(L), Tr_M(N)$ are independent. Hence $Tr_M(L) \cap Tr_M(N) = 0$.

Since $0 \ne L \subseteq Tr_M(L) \,\&\, 0 \ne N \subseteq Tr_M(N) \,\&\, Tr_M(L) \cap Tr_M(N) = 0$, $0 \ne Tr_M(L) \ne M$. Taking into account that $Tr_M(L)$ is a fully invariant submodule of M, it is easily seen that

$\{B \le M | Tr_M(L) \le B\}$ is a non-trivial preradical filter of M, contrary to the fact that every preradical filter of M is trivial.

Since all minimal submodules of M are isomorphic, M has exactly one homogeneous component.

Arguing similarly as in Example 1 [5], we obtain that M is finitely generated.

($\Leftarrow$) Assume that M is a finitely generated semisimple module and all minimal submodules of M are isomorphic. Arguing similarly as in Theorem 4 [5], we obtain that every preradical filter of M is trivial.

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