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Paley-Wiener-type theorems for the Hankel transform of positive half-integer order in weighted L^2-spaces

Abstract. In this paper, we obtain several analogues of the Paley-Wiener theorem for the Hankel transform of order $m+1/2$ with $m \in \mathbb{N}$ in weighted spaces $L^2((0;1);x^{2m}dx)$. These Paley-Wiener-type theorems describe a certain class of odd entire functions of exponential type $\sigma \le 1$ under this transformation and provide methods to constructing such a kind of Hankel transforms in terms of solutions from the corresponding spaces of some differential equations.

Key words: Paley-Wiener theorem, Hankel transform, Bessel function, Fubini's theorem, entire function of exponential type, half-integer order, weighted space

Анотація. У даній роботі отримано деякі аналоги теореми Пелі-Вінера для перетворення Ганкеля порядку $m+1/2$ з $m \in \mathbb{N}$ у вагових просторах $L^2((0;1);x^{2m}dx)$. Ці теореми типу Пелі-Вінера описують певний клас непарних цілих функцій експоненційного типу $\sigma \le 1$ при цьому перетворенні та вказують методи побудови такого роду перетворень Ганкеля в термінах розв'язків з відповідних просторів деяких диференціальних рівнянь.

Ключові слова: теорема Пелі-Вінера, перетворення Ганкеля, функція Бесселя, теорема Фубіні, ціла функція експоненційного типу, півцілий порядок, ваговий простір

MSC2020: Pri 33C10, 44A15, 46E30, Sec 30D10, 46F12

1. Introduction

Let $L^2(X)$ be the space of all measurable functions $f : X \to \mathbb{C}$ on a measurable set $X \subseteq \mathbb{R}$ with the norm

$$\|f\|_{L^2(X)} := \left(\int_X |f(x)|^2\, dx\right)^{1/2} < +\infty,$$

let $\gamma \in \mathbb{R}$ and let $L^2((0;1);x^\gamma dx)$ be the weighted Lebesgue space of all measurable functions $f:(0;1)\to\mathbb{C}$, satisfying

$$\int_0^1 x^\gamma |f(x)|^2\,dx < +\infty.$$

The function h belongs to the space $L^2((0;1);t^\gamma dt)$ if and only if the function $q(t)=t^{\gamma/2}h(t)$ belongs to $L^2(0;1)$.

An entire function G is said to be of ([19, p. 4], [21, p. 4262]) exponential type $\sigma\in[0;+\infty)$ if for any $\varepsilon>0$ there exists a constant $c(\varepsilon)$ such that $|G(z)|\le c(\varepsilon)\exp((\sigma+\varepsilon)|z|)$ for all $z\in\mathbb{C}$. The classical Paley-Wiener theorem is a fundamental result in the complex analysis that gives a description of the class of entire functions of exponential type that are square-integrable on the real axis. Let us denote by PW_σ^2 the class of all entire functions of exponential type $\sigma\in(0;+\infty)$ whose narrowing on $\mathbb{R}$ belongs to $L^2(\mathbb{R})$, and by $PW_{\sigma,-}^2$ we denote the class of odd entire functions from PW_σ^2. By the Paley-Wiener theorem (see [19, p. 69], [38, p. 13]), the class PW_σ^2 coincides with the class of functions G representable as

$$G(z)=\int_{-\sigma}^{\sigma} e^{itz}g(t)\,dt,\quad g\in L^2(-\sigma;\sigma),$$

and the class $PW_{\sigma,-}^2$ consists of the functions G of the form

$$G(z)=\int_0^{\sigma}\sin(tz)g(t)\,dt,\quad g\in L^2(0;\sigma).$$

Moreover, $\|g\|_{L^2(0;\sigma)}=\sqrt{2/\pi}\|G\|_{L^2(0;+\infty)}$ and

$$g(t)=\frac{2}{\pi}\int_0^{+\infty}G(z)\sin(tz)\,dz.$$

Let ([36, p. 40])

$$J_\nu(z)=\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)},\quad z\in\mathbb{C},$$

be the Bessel function of the first kind of index $\nu\in\mathbb{R}$, where Γ is the classical Gamma function. Since (see [36, p. 54])

$$J_{m+1/2}(z)=(-1)^m\sqrt{\frac{2}{\pi}}z^{m+1/2}\left(\frac{d}{zdz}\right)^m\left(\frac{\sin z}{z}\right),\quad m\in\mathbb{N},$$

the function $f(t)=z^m\sqrt{tz}J_{m+1/2}(tz)$ belongs to the space $L^2((0;1);t^{2m}dt)$ for every $m\in\mathbb{N}$ and $z\in\mathbb{C}$.

We define the Hankel transform $H_\nu \in L^2(0;+\infty)$ of order $\nu \geq -1/2$ of a function $h \in L^2(0;+\infty)$ by the formula (see [24, 26])

$$H_\nu(z) = \int_0^{+\infty} \sqrt{tz} J_\nu(tz) h(t)\, dt, \quad z \in [0;+\infty),$$

where the integral is taken in the L^2-sense or in the mean, that is,

$$\lim_{u\to+\infty} \int_0^{+\infty} \left| H_\nu(z) - \int_0^u \sqrt{tz} J_\nu(tz) h(t)\, dt \right|^2 dz = 0.$$

The Plancherel theorem for the Hankel transform (see [5, 6, 24, 26]) states that $\|H_\nu\|_{L^2(0;+\infty)} = \|h\|_{L^2(0;+\infty)}$ and

$$h(t) = \int_0^{+\infty} \sqrt{tz} J_\nu(tz) H_\nu(z)\, dz, \quad \nu > -1.$$

Due to the fundamental role of the Paley-Wiener theorem in harmonic and complex analysis, mathematicians have been seeking for generalizations of the classical results in various aspects (see the papers [1, 2, 3, 4, 5, 6, 7, 8, 19, 20, 21, 24, 25, 26, 27, 28, 36, 37, 38, 39] and bibliography in them). Since the Hankel transforms are generalizations of the Fourier transforms, it is natural to ask whether such a representation for entire functions is possible in this case also. An analogue of the Paley-Wiener theorem for Hankel transforms of order $\nu \geq -1/2$ in L^2-spaces was established in the papers [2, 4, 5, 6, 8, 26, 27, 28, 37, 39]. In particular, the following statement has been obtained by Griffith [8] (see also [2, 28, 39]).

Theorem 1 ([8]). *Let $\nu \geq -1/2$. A function G has the representation*

$$G(z) = z^{-\nu} \int_0^1 \sqrt{t} J_\nu(zt) g(t)\, dt$$

with $g \in L^2(0;1)$ if and only if it is an even entire function of exponential type $\sigma \leq 1$ such that $z^{\nu+1/2}G(z) \in L^2(0;+\infty)$.

The result of Griffith has been extended and generalized in different directions by Abreu [1], Andersen and de Jeu [4, 5], Betancor, Linares and Méndez [6], Pathak and Chaurasia [20], Trimèche [25], Tuan and Zayed [26, 27], Unni [28], Weiss [37], Zemanian [39], Vynnyts'kyi and the author [9, 10, 30]. The following theorem is valid.

Theorem 2 ([9, 10, 30]). *Let $\nu \geq 1/2$ and $p \in \mathbb{R}$. An entire function Ω has the representation*

$$\Omega(z) = z^{-\nu} \int_0^1 \sqrt{t} J_\nu(tz) t^{p-1} h(t)\, dt$$

with some function $h \in L^2((0;1);x^{2p}dx)$ *if and only if it is an even entire function of exponential type* $\sigma \le 1$ *such that* $z^{-\nu+1/2}(z^{2\nu}\Omega(z))' \in L^2(0;+\infty)$. *In this case,*

$$h(t) = t^{-p}\int_0^{+\infty} \sqrt{tz}J_{\nu-1}(tz)z^{-\nu+1/2}(z^{2\nu}\Omega(z))'\,dz.$$

Theorems 1 and 2 have been used in the study of completeness, minimality and basicity of the system $\left\{\sqrt{x\rho_k}J_\nu(x\rho_k) : k \in \mathbb{N}\right\}$ with $\nu \ge -1/2$ in $L^2(0;1)$ and more general system $\left\{x^{-p-1}\sqrt{x\rho_k}J_\nu(x\rho_k) : k \in \mathbb{N}\right\}$ with $\nu \ge 1/2$ and $p \in \mathbb{R}$ in the space $L^2((0;1);x^{2p}dx)$, where $(\rho_k)_{k\in\mathbb{N}}$ is an arbitrary sequence of distinct nonzero complex numbers such that $\rho_k^2 \ne \rho_n^2$ for $k \ne n$ (for details, see [9, 10, 30, 31, 32, 33]). Those results are formulated in terms of sequences of zeros of functions from certain classes of entire functions.

It is an actual problem to perform similar investigations in an important special cases where $\nu = \pm(m+1/2)$ with $m \in \mathbb{N}$. This situation differs from the other cases by the fact that the Bessel function $J_\nu(z)$ is expressible in finite terms by means of algebraic and trigonometrical functions of z. Approximation properties (completeness, minimality and basicity) of the systems of Bessel functions of index $\nu < -1$, $\nu \notin \mathbb{Z}$, in weighted L^2-spaces were investigated in the papers [11, 12, 13, 14, 15, 16, 17, 18, 22, 23, 29, 34, 35]. In this case, in [17] (see also [11, 14, 15, 16, 18]) were proved some analogues of the Paley-Wiener theorem for the Hankel type transform of order $-m-1/2$, $m \in \mathbb{N}$, in weighted spaces $L^2((0;1);x^{2m}dx)$. These Paley-Wiener-type theorems describe under this transform a certain class of even entire functions Q of exponential type $\sigma \le 1$ of the form

$$Q(z) = z^m\int_0^1 \sqrt{tz}J_{-m-1/2}(tz)t^{2m}h(t)\,dt, \quad h \in L^2((0;1);x^{2m}dx),$$

in terms of solutions of some differential equations. Such a class of Hankel type transforms arises in the investigation of some non-classical boundary value problems for Bessel equation (see [12, 13, 22, 23, 29, 34, 35]) for which the system of their canonical eigenfunctions is over-complete.

The aim of this paper is to establish the Paley-Wiener-type theorems for the Hankel transform (2.1) of order $m+1/2$ with $m \in \mathbb{N}$ in weighted spaces $L^2((0;1);x^{2m}dx)$ (see Theorems 3–5). Theorem 3 gives a description of a certain class of odd entire functions of exponential type $\sigma \le 1$ under this transformation. Theorems 4 and 5 provide a method to constructing the Hankel transform (2.1) in terms of solutions from the corresponding spaces of certain differential equations. This complements the results of papers [9, 10, 11, 15, 17, 18, 30, 31, 32, 33].

2. Main results

Here and subsequently, by $c_1, c_2, \dots$ we denote arbitrary positive constants. Let $[x]$ be the integer part of a real number x, and let $\left(z^{-1}d/dz\right)^m$ be the m-th power of the differential operator $z^{-1}d/dz$.

Our main results are the following statements.

Theorem 3. *Let $m \in \mathbb{N}$. An entire function $H_{m+1/2}$ has the representation*

$$H_{m+1/2}(z) = z^m \int_0^1 \sqrt{tz} J_{m+1/2}(tz) t^{2m} h_{m+1/2}(t)\, dt \tag{2.1}$$

with some function $h_{m+1/2} \in L^2((0;1); x^{2m}dx)$ if and only if it is an odd entire function of exponential type $\sigma \le 1$ such that

$$H_{m+1/2}(0) = 0, \tag{2.2}$$

$$\left(\frac{1}{z}\frac{d}{dz}\right)^s H_{m+1/2}(z)\bigg|_{z=0} = 0, \quad s \in \{1; 2; \dots; m-1\}, \tag{2.3}$$

and the function $\left(z^{-1}d/dz\right)^m H_{m+1/2}(z)$ belongs to the space $PW^2_{1,-}$. If these conditions are fulfilled, then

$$h_{m+1/2}(t) = \sqrt{\frac{2}{\pi}} \frac{1}{t^{3m}} \int_0^{+\infty} \sin(tz) \left(\frac{1}{z}\frac{d}{dz}\right)^m H_{m+1/2}(z)\, dz. \tag{2.4}$$

Proof. *Necessity.* Let $m \in \mathbb{N}$ and $H_{m+1/2}$ has the representation (2.1) with some function $h_{m+1/2} \in L^2((0;1); x^{2m}dx)$. Then

$$H_{m+1/2}(z) = \int_0^1 (tz)^{m+1/2} J_{m+1/2}(tz) t^m h_{m+1/2}(t)\, dt.$$

Since

$$z^\nu J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2\nu+2k}}{2^{\nu+2k} k! \Gamma(\nu+k+1)}, \quad \nu \in \mathbb{R}, \tag{2.5}$$

the function $H_{m+1/2}$ is an odd entire function and we have (2.2). Further, since ([30])

$$|\sqrt{z} J_\nu(z)| \le c_1 e^{|\operatorname{Im} z|} \left(\frac{|z|}{1+|z|}\right)^{\nu+1/2}, \quad z \in \mathbb{C}, \quad \nu \ge -1/2,$$

applying Schwartz's inequality, for all $z \in \mathbb{C}$ we get $(q_{m+1/2}(t) := t^m h_{m+1/2}(t) \in L^2(0;1))$

$$\begin{aligned} |H_{m+1/2}(z)| &\le \|q_{m+1/2}\|_{L^2(0;1)} |z|^m \left(\int_0^1 \left|\sqrt{tz} J_{m+1/2}(tz)\right|^2 t^{2m}\, dt\right)^{1/2} \\ &\le c_2 |z|^{2m+1} e^{|\operatorname{Im} z|} \left(\int_0^1 \frac{t^{4m+2}}{(1+t|z|)^{2m+2}}\, dt\right)^{1/2} \\ &\le c_3 \frac{e^{|\operatorname{Im} z|}}{\sqrt{1+|\operatorname{Im} z|}} (1+|z|)^m, \quad m \in \mathbb{N}. \end{aligned}$$

Furthermore, $H_{m+1/2}$ is an entire function of exponential type $\sigma \le 1$. Besides, using relations (see [36, pp. 46, 54])

$$\left(\frac{1}{z}\frac{\partial}{\partial z}\right)^s (tz)^\nu J_\nu(tz) = t^{2s}(tz)^{\nu-s}J_{\nu-s}(tz), \quad t,\nu\in\mathbb{R}, \quad s\in\mathbb{N},$$

$$\sqrt{z}J_{1/2}(z) = \sqrt{\frac{2}{\pi}}\sin z,$$

we obtain

$$\left(\frac{1}{z}\frac{d}{dz}\right)^s H_{m+1/2}(z) = \int_0^1 (tz)^{m-s+1/2}J_{m-s+1/2}(tz)t^{2s+m}h_{m+1/2}(t)\,dt, \quad (2.6)$$

for $s\in\{1;2;\ldots;m\}$, and

$$\begin{aligned}\left(\frac{1}{z}\frac{d}{dz}\right)^m H_{m+1/2}(z) &= \int_0^1 \sqrt{tz}J_{1/2}(tz)t^{3m}h_{m+1/2}(t)\,dt \\ &= \sqrt{\frac{2}{\pi}}\int_0^1 t^{2m}\sin(tz)q_{m+1/2}(t)\,dt,\end{aligned} \quad (2.7)$$

where $q_{m+1/2}(t) = t^m h_{m+1/2}(t)$ and $q_{m+1/2}\in L^2(0;1)$. Using (2.5) and (2.6), we get (2.3). In addition, according to the Paley-Wiener theorem, the function $\left(z^{-1}d/dz\right)^m H_{m+1/2}(z)$ belongs to the space $PW^2_{1,-}$. Therefore, the necessity has been proved.

Sufficiency. If all the conditions of the theorem hold, then from the formula for the inverse Fourier sine transformation it follows that the function

$$q_{m+1/2}(t) = \sqrt{\frac{2}{\pi}}\frac{1}{t^{2m}}\int_0^{+\infty}\sin(tz)\left(\frac{1}{z}\frac{d}{dz}\right)^m H_{m+1/2}(z)\,dz$$

belongs to the space $L^2(0;1)$, and

$$\left(\frac{1}{z}\frac{d}{dz}\right)^m H_{m+1/2}(z) = \int_0^1 \sqrt{tz}J_{1/2}(tz)t^{3m}h_{m+1/2}(t)\,dt, \quad m\in\mathbb{N},$$

where $h_{m+1/2}(t) = t^{-m}q_{m+1/2}(t) \in L^2((0;1);x^{2m}dx)$. Therefore, we have (2.4). Further, taking into account (2.3) and consecutively applying the Fubini theorem $m-1$ times, we obtain

$$H'_{m+1/2}(z) = \int_0^1 z(tz)^{m-1/2}J_{m-1/2}(tz)t^{m+2}h_{m+1/2}(t)\,dt.$$

Using equalities (2.5) and $((tw)^\nu J_\nu(tw))'_w = t(tw)^\nu J_{\nu-1}(tw)$, $t,\nu\in\mathbb{R}$ (see [36, p. 45]), we have $\int_0^z t(wt)^\nu J_{\nu-1}(wt)\,dw = (tz)^\nu J_\nu(tz)$. Furthermore, by using (2.2) and applying Fubini's theorem, we get

$$\begin{aligned}H_{m+1/2}(z) &= \int_0^1 t^{2+m}h_{m+1/2}(t)\,dt\int_0^z w(tw)^{m-1/2}J_{m-1/2}(tw)\,dw \\ &= \int_0^1 t^m h_{m+1/2}(t)\,dt\int_0^z t(tw)^{m+1/2}J_{m-1/2}(tw)\,dw \\ &= z^m\int_0^1 \sqrt{tz}J_{m+1/2}(tz)t^{2m}h_{m+1/2}(t)\,dt.\end{aligned}$$

Hence, the function $H_{m+1/2}$ can be represented in the form (2.1). Thus, the theorem is proved.

Example 1. Let $m = 1$. The function

$$H_{3/2}(z) = \frac{z^2 \sin z}{\pi^2(z^2 - \pi^2)} \tag{2.8}$$

admits the representation (2.1) with

$$h_{3/2}(t) = \sqrt{\frac{\pi}{2}} \frac{1}{\pi^3 t^3} \left(\pi t \cos(\pi t) - \sin(\pi t)\right). \tag{2.9}$$

Indeed, the function $H_{3/2}(z)$ is an odd entire function of exponential type $\sigma \leq 1$ such that $H_{3/2}(0) = 0$,

$$\frac{H'_{3/2}(z)}{z} = \frac{z \cos z}{\pi^2(z^2 - \pi^2)} - \frac{2 \sin z}{(z^2 - \pi^2)^2},$$

and the function $z^{-1} H'_{3/2}(z)$ belongs to the space $PW^2_{1,-}$. In addition, by formula (2.4), we obtain

$$\begin{aligned} h_{3/2}(t) &= \sqrt{\frac{2}{\pi}} \frac{1}{t^3} \int_0^{+\infty} \frac{H'_{3/2}(z)}{z} \sin(tz)\, dz \\ &= \frac{1}{\sqrt{2\pi} t^3} \int_{-\infty}^{+\infty} \left(\frac{z \cos z}{\pi^2(z^2 - \pi^2)} - \frac{2 \sin z}{(z^2 - \pi^2)^2} \right) \sin(tz)\, dz \\ &= \sqrt{\frac{\pi}{2}} \frac{1}{\pi^3 t^3} \left(\pi t \cos(\pi t) - \sin(\pi t)\right). \end{aligned}$$

Hence, according to Theorem 3, the function (2.8) can be represented in the form (2.1) with the function (2.9) for $m = 1$.

Theorem 4. *Let $m \in \mathbb{N}$. An entire function $H_{m+1/2}$ has the representation (2.1) with some function $h_{m+1/2} \in L^2((0;1); x^{2m}dx)$ if and only if the differential equation*

$$(2m - 1)f(z) - zf'(z) = H_{m+1/2}(z) \tag{2.10}$$

has a solution $f = G_{m-1/2}$ which can be presented in the form

$$G_{m-1/2}(z) = z^{m-1} \int_0^1 \sqrt{tz} J_{m-1/2}(tz) t^{2m-1} h_{m+1/2}(t)\, dt. \tag{2.11}$$

Proof. *Necessity.* Let $m \in \mathbb{N}$ and the function $H_{m+1/2}$ admits representation (2.1) with some function $h_{m+1/2} \in L^2((0;1); x^{2m}dx)$. Since (see [36, p. 45])

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z), \quad \nu \in \mathbb{R}, \tag{2.12}$$

we have

$$\begin{aligned}H_{m+1/2}(z) &= z^m \int_0^1 \sqrt{tz} J_{m+1/2}(tz) t^{2m} h_{m+1/2}(t)\,dt \\ &= z^m \int_0^1 \sqrt{tz} \left(\frac{2(m-1/2) J_{m-1/2}(tz)}{tz} - J_{m-3/2}(tz) \right) t^{2m} h_{m+1/2}(t)\,dt \\ &= (2m-1) \int_0^1 (tz)^{m-1/2} J_{m-1/2}(tz) t^m h_{m+1/2}(t)\,dt \\ &- z^{m+1/2} \int_0^1 J_{m-3/2}(tz) t^{2m+1/2} h_{m+1/2}(t)\,dt.\end{aligned}$$

Let

$$G_{m-1/2}(z) := \int_0^1 (tz)^{m-1/2} J_{m-1/2}(tz) t^m h_{m+1/2}(t)\,dt.$$

Then

$$G_{m-1/2}(z) = z^{m-1} \int_0^1 \sqrt{tz} J_{m-1/2}(tz) t^{2m-1} h_{m+1/2}(t)\,dt.$$

Since (see [36, p. 45])

$$(z^\nu J_\nu(z))' = z^\nu J_{\nu-1}(z), \quad \nu \in \mathbb{R}, \tag{2.13}$$

we get

$$G'_{m-1/2}(z) = \int_0^1 (tz)^{m-1/2} J_{m-3/2}(tz) t^{m+1} h_{m+1/2}(t)\,dt.$$

Therefore, we obtain $(2m-1)G_{m-1/2} - zG'_{m-1/2}(z) = H_{m+1/2}(z)$, that is, the function (2.11) is a solution to the equation (2.10). The necessity has been proved.

Sufficiency. Let $m \in \mathbb{N}$ and the equation (2.10) has a solution $f = G_{m-1/2}$ representing in the form (2.11) with some function $h_{m+1/2} \in L^2((0;1); x^{2m}dx)$. We have

$$\begin{aligned}f(z) &= z^{m-1} \int_0^1 \sqrt{tz} J_{m-1/2}(tz) t^{2m-1} h_{m+1/2}(t)\,dt \\ &= \int_0^1 (tz)^{m-1/2} J_{m-1/2}(tz) t^m h_{m+1/2}(t)\,dt.\end{aligned}$$

Using (2.12) and (2.13), we obtain

$$\begin{aligned}H_{m+1/2}(z) &= (2m-1) f(z) - z f'(z) \\ &= (2m-1) \int_0^1 (tz)^{m-1/2} J_{m-1/2}(tz) t^m h_{m+1/2}(t)\,dt \\ &- z \int_0^1 (tz)^{m-1/2} J_{m-3/2}(tz) t^{m+1} h_{m+1/2}(t)\,dt \\ &= z^m \int_0^1 \sqrt{tz} \left((2m-1) \frac{J_{m-1/2}(tz)}{tz} - J_{m-3/2}(tz) \right) t^{2m} h_{m+1/2}(t)\,dt \\ &= z^m \int_0^1 \sqrt{tz} J_{m+1/2}(tz) t^{2m} h_{m+1/2}(t)\,dt.\end{aligned}$$

Hence, the function $H_{m+1/2}$ has the form (2.1) with some function $h_{m+1/2} \in L^2((0;1);x^{2m}dx)$. Theorem 4 is proved.

Remark 1. Theorem 4 provides a method for constructing an integral representation of the class of entire functions $H_{m+1/2}$ of the form (2.1) for an arbitrary $m \in \mathbb{N}$ if the solution to the equation (2.10) is known for some m.

Example 2. Let $m = 2$. The function

$$H_{5/2}(z) = z^2 \frac{-z\cos z + 3\sin z}{\pi^2(z^2-\pi^2)} + \frac{2z^2 \sin z}{(z^2-\pi^2)^2} \tag{2.14}$$

has the representation (2.1) with

$$h_{5/2}(t) = \sqrt{\frac{\pi}{2}} \frac{1}{\pi^3 t^4} \left(\pi t \cos(\pi t) - \sin(\pi t)\right). \tag{2.15}$$

Indeed, according to Example 1, the function $G_{3/2}(z) = H_{3/2}(z)$ admits the representation (2.11) with $m = 2$ and $h_{5/2}(t) = t^{-1}h_{3/2}(t)$, where the function $H_{3/2}$ is defined by the formula (2.8) and $h_{3/2}$ is determined by the equality (2.9). In addition, the function $G_{3/2}$ is a solution to the equation (2.10) for $m = 2$, because

$$G'_{3/2}(z) = \frac{z^2\cos z}{\pi^2(z^2-\pi^2)} - \frac{2z\sin z}{(z^2-\pi^2)^2}.$$

Thus, by Theorem 4, the function (2.14) can be represented in the form (2.1) with $m = 2$ and the function (2.15).

Theorem 5. *Let $m \in \mathbb{N}$. An entire function $H_{m+1/2}$ has the representation (2.1) with some function $h_{m+1/2} \in L^2((0;1);x^{2m}dx)$ if and only if the differential equation*

$$\begin{aligned}(-1)^{m-1}\sqrt{\frac{2}{\pi}}\Bigg(&-\sum_{s=0}^{[m/2]} \frac{(m+2s)!}{(2s)!(m-2s)!2^{2s}} z^{m-2s} f^{(m-2s)}(z) \\ &+ \sum_{s=0}^{[(m-1)/2]} \frac{(m+2s+1)!}{(2s+1)!(m-2s-1)!2^{2s+1}} z^{m-2s-1} f^{(m-2s-1)}(z)\Bigg) \\ &= H_{m+1/2}(z)\end{aligned} \tag{2.16}$$

has a solution $f = F_{m+1/2}$ belonging to $PW^2_{1,-}$. Moreover, the function $\left(z^{-1}d/dz\right)^m H_{m+1/2}(z)$ also belongs to the space $PW^2_{1,-}$ and $h_{m+1/2}$ can be found by the formula (2.4) and the formula

$$h_{m+1/2}(t) = \frac{2}{\pi}\frac{1}{t^m}\int_0^{+\infty} F_{m+1/2}(z)\sin(tz)\,dz. \tag{2.17}$$

Proof. *Necessity.* Let the function $H_{m+1/2}$ admits the representation (2.1) with some function $h_{m+1/2} \in L^2((0;1);x^{2m}dx)$, $m \in \mathbb{N}$, and

$$F_{m+1/2}(z) = \int_0^1 \sin(tz) q_{m+1/2}(t)\, dt, \tag{2.18}$$

where $q_{m+1/2}(t) := t^m h_{m+1/2}(t) \in L^2(0;1)$. Then, we have (2.6) and (2.7). According to the Paley-Wiener theorem, the functions $F_{m+1/2}(z)$ and $\left(z^{-1}d/dz\right)^m H_{m+1/2}(z)$ belong to $PW^2_{1,-}$. Further, since

$$F^{(m-2s)}_{m+1/2}(z) = (-1)^{s+m} \int_0^1 t^{m-2s} \sin\left(tz - \frac{m\pi}{2}\right) q_{m+1/2}(t)\, dt,$$

for $s \in \{0;1;\ldots;[m/2]\}$,

$$F^{(m-2s-1)}_{m+1/2}(z) = (-1)^{s+m+1} \int_0^1 t^{m-2s-1} \cos\left(tz - \frac{m\pi}{2}\right) q_{m+1/2}(t)\, dt,$$

for $s \in \{0;1;\ldots;[(m-1)/2]\}$, and (see [36, p. 53])

$$\begin{aligned}
&\sqrt{\frac{\pi}{2}}(tz)^{m+1/2} J_{m+1/2}(tz) \\
&= \sin\left(tz - \frac{m\pi}{2}\right) \sum_{s=0}^{[m/2]} \frac{(-1)^s (m+2s)!}{(2s)!(m-2s)!2^{2s}} (zt)^{m-2s} \\
&+ \cos\left(tz - \frac{m\pi}{2}\right) \sum_{s=0}^{[(m-1)/2]} \frac{(-1)^s (m+2s+1)!}{(2s+1)!(m-2s-1)!2^{2s+1}} (zt)^{m-2s-1}, \quad m \in \mathbb{N},
\end{aligned}$$

we obtain

$$\begin{aligned}
&(-1)^{m-1}\sqrt{\frac{2}{\pi}} \left(- \sum_{s=0}^{[m/2]} \frac{(m+2s)!}{(2s)!(m-2s)!2^{2s}} z^{m-2s} F^{(m-2s)}_{m+1/2}(z) \right. \\
&\left. + \sum_{s=0}^{[(m-1)/2]} \frac{(m+2s+1)!}{(2s+1)!(m-2s-1)!2^{2s+1}} z^{m-2s-1} F^{(m-2s-1)}_{m+1/2}(z) \right) \\
&= \sqrt{\frac{2}{\pi}} \int_0^1 \left(\sin\left(tz - \frac{m\pi}{2}\right) \sum_{s=0}^{[m/2]} \frac{(-1)^s (m+2s)!}{(2s)!(m-2s)!2^{2s}} (zt)^{m-2s} \right. \\
&\left. + \cos\left(tz - \frac{m\pi}{2}\right) \sum_{s=0}^{[(m-1)/2]} \frac{(-1)^s (m+2s+1)!}{(2s+1)!(m-2s-1)!2^{2s+1}} (zt)^{m-2s-1} \right) q_{m+1/2}(t)\, dt \\
&= \int_0^1 (tz)^{m+1/2} J_{m+1/2}(tz) t^m h_{m+1/2}(t)\, dt \\
&= z^m \int_0^1 \sqrt{tz} J_{m+1/2}(tz) t^{2m} h_{m+1/2}(t)\, dt = H_{m+1/2}(z).
\end{aligned}$$

Therefore, the necessity has been proved.

Sufficiency. Let the equation (2.16) has a solution $f = F_{m+1/2}$ belonging to $PW^2_{1,-}$. Due to the Paley-Wiener theorem, we have

$$f(z) = \int_0^1 \sin(tz) q_{m+1/2}(t)\,dt, \quad q_{m+1/2} \in L^2(0;1).$$

Hence, as above, we get

$$\begin{aligned} H_{m+1/2}(z) = (-1)^{m-1}\sqrt{\frac{2}{\pi}} \Bigg(&- \sum_{s=0}^{[m/2]} \frac{(m+2s)!}{(2s)!(m-2s)!2^{2s}} z^{m-2s} f^{(m-2s)}(z) \\ &+ \sum_{s=0}^{[(m-1)/2]} \frac{(m+2s+1)!}{(2s+1)!(m-2s-1)!2^{2s+1}} z^{m-2s-1} f^{(m-2s-1)}(z) \Bigg) \\ = z^m \int_0^1 \sqrt{tz} J_{m+1/2}(tz) t^{2m} h_{m+1/2}(t)\,dt, \end{aligned}$$

where $h_{m+1/2}(t) := t^{-m} q_{m+1/2}(t) \in L^2((0;1); x^{2m}dx)$. Thus, we obtain representation (2.1). Formula (2.4) is proved in Theorem 3, and the formula (2.17) follows from the equality (2.18) and the formula for the inverse Fourier sine transformation. Theorem 5 is proved.

Example 3. The function $H_{7/2}(z) = \sqrt{2/\pi} z^7 \cos z$ cannot be represented in the form (2.1) with $m = 3$. In fact, for this function $H_{7/2}$ and $m = 3$ the differential equation (2.16) has the form

$$-z^3 f'''(z) + 6z^2 f''(z) - 15 z f'(z) + 15 f(z) = z^7 \cos z,$$

and its solution is the function $F_{7/2}(z) = z^2(z^2 - 15)\sin z + 3z(2z^2 - 5)\cos z + C_1 z + C_2 z^3 + C_3 z^5$. But there are no constants C_1, C_2 and C_3 for which the function $F_{7/2}$ belongs to $PW^2_{1,-}$ since $F_{7/2} \notin L^2(\mathbb{R})$. Therefore, the equation (2.16) with $H_{7/2}(z) = \sqrt{2/\pi} z^7 \cos z$ and $m = 3$ has no solution belonging to $PW^2_{1,-}$. Hence, by Theorem 5, the function $H_{7/2}$ cannot be represented in the form (2.1) with $m = 3$.

Remark 2. Analogues of Theorems 3–5 for the Hankel type transform of order $-m-1/2$ with $m \in \mathbb{N}$ in weighted spaces $L^2((0;1); x^{2m}dx)$ were obtained in [17, 18].

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