УДК 517.5+517.9

B. V. Vynnyts'kyi, R. V. Khats'

SOME APPROXIMATION PROPERTIES OF THE SYSTEMS OF BESSEL FUNCTIONS OF INDEX $-3/2$

B. V. Vynnyts'kyi, R. V. Khats'. *Some approximation properties of the systems of Bessel functions of index* $-3/2$, Mat. Stud. **34** (2010), 152–159.

We investigate an approximation properties of the systems of Bessel functions of index $-3/2$.

Б. В. Винницкий, Р. В. Хаць. *Некоторые аппроксимационные свойства систем функций Бесселя с индексом* $-3/2$ // Мат. Студiї. – 2010. – Т.34, №2. – C.152–159.

Исследуются аппроксимационные свойства систем функций Бесселя с индексом $-3/2$.

Let $L^2((0;1);x^2dx)$ be the space of functions $f\colon (0;1)\to\mathbb{C}$ such that $tf(t)\in L^2(0;1)$ with the inner product $\langle f_1;f_2\rangle = \int_0^1 t^2 f_1(t)\overline{f_2(t)}\,dt$ and the norm $\|f\| = \sqrt{\int_0^1 t^2|f(t)|^2\,dt}$. Let $J_\nu(x)=\sum_{k=0}^\infty \dfrac{(-1)^k(x/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}$ be Bessel's function of the first kind of index ν. It is well known (see, for instance, [1–3]) that the function J_ν is a solution of the equation $y''+y'/x+(1-\nu^2/x^2)y=0$, the function $y(x)=J_\nu(xs)$ is a solution of the equation $y''+y'/x-y\nu^2/x^2=-s^2y$, and the function $y(x)=\sqrt{xs}J_\nu(xs)$ satisfies the equation

$$y''-\frac{\nu^2-1/4}{x^2}y=-s^2y. \tag{1}$$

The function $J_{-3/2}$ has [1–3] an infinite set $(s_k\colon k\in\mathbb{Z}\setminus\{0\})$ of zeros, among them s_1 and $s_{-1}=\overline{s_1}=-s_1$ are two purely imaginary zeros, positive zeros s_k, $k\in\mathbb{N}\setminus\{1\}$, and negative zeros $s_{-k}:=-s_k$, $k\in\mathbb{N}\setminus\{1\}$. Moreover, [1–3], $\sqrt{z}J_{-3/2}(z)=-\sqrt{\dfrac{2}{\pi}}z^{-1}(\cos z+z\sin z)$. The function $s\sqrt{xs}J_{-3/2}(xs)$ belongs to the space $L^2((0;1);x^2dx)$ for every $s\in\mathbb{C}$.

The problem of completeness of the system $(\sqrt{xs_k}J_\nu(xs_k)\colon k\in\mathbb{N})$ for $\nu\notin\mathbb{Z}$, $\nu<0$, were studied by R. Boas, H. Pollard, O. Shavala, B. Vynnyts'kyi, V. Dilnyi (see [4–8]) and numerous other mathematicians. When studying a boundary value problem [5–7] generated by equation (1), in [5] the following statement was proved.

Theorem A. *Let $(s_k\colon k\in\mathbb{Z}\setminus\{0\})$, $s_{-k}=-s_k$, be a sequence of zeros of the function $J_{-3/2}$. Then the system $(s_k\sqrt{xs_k}J_{-3/2}(xs_k)\colon k\in\mathbb{N})$ is complete in the space $L^2((0;1);x^2dx)$. The system $(s_k\sqrt{xs_k}J_{-3/2}(xs_k)\colon k\in\mathbb{N}\setminus\{1\})$ is complete, minimal and is not a basis in the space $L^2((0;1);x^2dx)$. There exists a biorthogonal system $(g_k\colon k\in\mathbb{N}\setminus\{1\})$ in this space which is formed by the functions g_k, defined by the formula*

$$\overline{g}_k(t)=\frac{\pi(1+s_k^2)}{s_k^4t^2}(s_k\sqrt{ts_k}J_{-3/2}(ts_k)-s_1\sqrt{ts_1}J_{-3/2}(ts_1)).$$

2000 *Mathematics Subject Classification:* 30D15, 30D20, 30E15, 42B10, 44A15.

Keywords: approximation property, Bessel function, system of functions
doi:10.30970/ms.34.2.152-159

Define for the function $f \in L^2((0;1);x^2dx)$ the numbers d_k by the equality

$$d_k = \int_0^1 t^2 f(t) g_k(t)\,dt.$$

Since a system $(s_k\sqrt{xs_k}J_{-3/2}(xs_k)\colon k \in \mathbb{N}\setminus\{1\})$ is not a basis of the space $L^2((0;1);x^2dx)$, not for each function $f \in L^2((0;1);x^2dx)$ the series $\sum_{k\in\mathbb{N}\setminus\{1\}} d_k s_k\sqrt{xs_k}J_{-3/2}(xs_k)$ converges in $L^2((0;1);x^2dx)$ to the function f. The question of restoration possibility of the function $f \in L^2((0;1);x^2dx)$ behind numbers d_k remained open. We give a positive answer to this question (see also [9]), proving the completeness of the system $(g_k\colon k \in \mathbb{N}\setminus\{1\})$.

Let $(s_k\colon k \in \mathbb{N})$ be an arbitrary sequence of nonzero complex numbers. First, we investigate an approximation properties of the system $(U_k\colon k \in \mathbb{N}\setminus\{1\})$, where

$$U_k(x) = \frac{s_k\sqrt{xs_k}J_{-3/2}(xs_k) - s_1\sqrt{xs_1}J_{-3/2}(xs_1)}{x^2(s_1^2 - s_k^2)}.$$

We say that, an entire function G is of formal exponential type $\sigma \in (0;+\infty)$ if

$$|G(z)| \le c(\varepsilon)\exp((\sigma+\varepsilon)|z|), \quad z \in \mathbb{C},$$

for each $\varepsilon > 0$ and some constant $c(\varepsilon)$. Denote by PW_σ^2 the set of all entire functions of formal exponential type $\sigma \in (0;+\infty)$ belonging to the space $L^2(\mathbb{R})$ on the real axis $\mathbb{R}$ in $\mathbb{C}$, and by $PW_{\sigma,+}^2$ we denote the class of even entire functions from PW_σ^2. According to the Paley-Wiener theorem [10–14], the class PW_σ^2 coincides with the class of functions G admitting the representation

$$G(z) = \int_{-\sigma}^{\sigma} e^{itz} g(t)\,dt, \quad g \in L^2(-\sigma;\sigma),$$

and the class $PW_{\sigma,+}^2$ consists of the functions G representing in the form

$$G(z) = \int_0^{\sigma} \cos(tz) g(t)\,dt, \quad g \in L^2(0;\sigma).$$

Moreover, $\|g\|_{L^2(0;\sigma)} = \sqrt{2/\pi}\|G\|_{L^2(0;+\infty)}$ and

$$g(t) = \frac{2}{\pi}\int_0^{+\infty} \cos(tz) G(z)\,dz.$$

Theorem 1. *Suppose that s_1 is an arbitrary complex number. An entire function Q can be represented in the form*

$$Q(z) = \int_0^1 (z\sqrt{tz}J_{-3/2}(tz) - s_1\sqrt{ts_1}J_{-3/2}(ts_1))h(t)\,dt, \quad h \in L^2((0;1);x^2dx), \tag{2}$$

if and only if Q is an even entire function, $Q(s_1) = 0$, $Q'(0) = 0$ and the function $Q'(z)/z$ belongs to the space $PW_{1,+}^2$. If these conditions hold then

$$h(t) = -\sqrt{\frac{2}{\pi}}\int_0^{+\infty} \frac{Q'(z)}{tz}\cos(tz)\,dz.$$

Proof. Let the function Q is representable in the form (2). Since

$$z\sqrt{tz}J_{-3/2}(tz) = -\sqrt{\frac{2}{\pi}}\frac{\cos(tz) + tz\sin(tz)}{t},$$

we obtain that $Q(z) = -\sqrt{\frac{2}{\pi}}\int_0^1 \frac{(\cos(tz) + tz\sin(tz)) - (\cos(ts_1) + ts_1\sin(ts_1))}{t}h(t)\,dt.$
Therefore, Q is an even entire function, $Q(s_1) = 0$ and

$$Q'(z) = -\sqrt{\frac{2}{\pi}} \int_0^1 tz \cos(tz) h(t)\, dt, \quad \frac{Q'(z)}{z} = -\sqrt{\frac{2}{\pi}} \int_0^1 \cos(tz) q(t)\, dt,$$

where $q(t) = th(t)$. Since $h \in L^2((0;1); x^2dx)$, then $q \in L^2(0;1)$ and, hence, according to Paley-Wiener theorem, the function $Q'(z)/z$ belongs to the space $PW^2_{1,+}$. Conversely, if all the conditions of the theorem hold then the function $q(t) = -\sqrt{2/\pi} \int_0^{+\infty} \frac{Q'(z)}{z} \cos(tz)\, dz$ belongs to the space $L^2(0;1)$ and $Q'(z) = -\sqrt{\frac{2}{\pi}} \int_0^1 z \cos(tz) q(t)\, dt$.

Using Fubini's theorem, we get

$$Q(z) = Q(z) - Q(s_1) = -\sqrt{\frac{2}{\pi}} \int_0^1 q(t)\, dt \int_{s_1}^z w \cos(tw)\, dw =$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^1 \frac{(\cos(tz) + tz\sin(tz)) - (\cos(ts_1) + ts_1\sin(ts_1))}{t} \frac{q(t)}{t}\, dt =$$

$$= \int_0^1 (z\sqrt{tz} J_{-3/2}(tz) - s_1\sqrt{ts_1} J_{-3/2}(ts_1)) h(t)\, dt,$$

where $h(t) = q(t)/t$. Since $q \in L^2(0;1)$, one has that $h \in L^2((0;1); x^2dx)$, and the proof of the theorem is completed. □

Let $\widetilde{E}_{2,+}(s_1)$ be the class of the entire functions Q that can be represented in the form (2), and let $E_{2,+}(s_1)$ be the class of nonzero even entire functions Q such that $Q(s_1) = 0$, $Q'(0) = 0$ and the function $Q'(z)/z$ belongs to the space $PW^2_{1,+}$.

Corollary 1. $\widetilde{E}_{2,+}(s_1) = E_{2,+}(s_1)$ *for every* $s_1 \in \mathbb{C}$.

Corollary 2. *The class* $E_{2,+}(s_1)$ *coincides with the set of the entire functions* Q *that can be represented in the form*

$$Q(z) = \sqrt{\frac{2}{\pi}} \int_0^1 \frac{(\cos(tz) + tz\sin(tz)) - (\cos(ts_1) + ts_1\sin(ts_1))}{t^2} q(t)\, dt, \quad q \in L^2(0;1). \quad (3)$$

Example 1. The function $Q(z) = \cos z$ belongs to $E_{2,+}(s_1)$ if $s_1 = \pi k + \pi/2$, $k \in \mathbb{Z}$.

Example 2. The function $Q(z) = \cos z + z\sin z$ does not belong to $E_{2,+}(s_1)$ for any $s_1 \in \mathbb{C}$.

Example 3. The function $Q(z) = \dfrac{\cos z + z\sin z}{z^2 - s_2^2}$ belongs to $E_{2,+}(s_1)$, if the numbers $s_1 \in \mathbb{C}$ and $s_2 \in \mathbb{C}$ are distinct zeros of the function $D(z) = \cos z + z\sin z$.

Theorem 2. *Let* $(s_k\colon k \in \mathbb{N})$ *be an arbitrary sequence of distinct nonzero complex numbers such that* $s_k^2 \neq s_n^2$ *if* $k \neq n$. *For a system* $(U_k\colon k \in \mathbb{N} \setminus \{1\})$ *to be incomplete in the space* $L^2((0;1); x^2dx)$ *it is necessary and sufficient that a sequence* $(s_k\colon k \in \mathbb{Z} \setminus \{0\})$, $s_{-k} := -s_k$ *is a subsequence of zeros of some nonzero function* $Q \in E_{2,+}(s_1)$.

Proof. Incompleteness of a system $(U_k\colon k \in \mathbb{N} \setminus \{1\})$ is equivalent to the incompleteness of the system $((s_1^2 - s_k^2)U_k\colon k \in \mathbb{N} \setminus \{1\})$. According to the well-known completeness criterion, the last system is incomplete in the space $L^2((0;1); x^2dx)$ if and only if there exists a nonzero function $h \in L^2((0;1); x^2dx)$ such that

$$\int_0^1 (s_k\sqrt{xs_k} J_{-3/2}(xs_k) - s_1\sqrt{xs_1} J_{-3/2}(xs_1)) h(x)\, dx = 0$$

for all $k \in \mathbb{N} \setminus \{1\}$. Hence, taking into account the previous theorem, we obtain the required proposition. Theorem 2 is proved. □

Lemma 1. *Let $s_1 \in \mathbb{C}$ be an arbitrary number, $q \in L^2(0;1)$ and*

$$I_1(z) = \sqrt{\frac{2}{\pi}} \int_0^{\frac{1}{2}} \frac{t(z-s_1)\sin(tz)}{t^2} q(t)\,dt, \quad I_2(z) = 2\sqrt{\frac{2}{\pi}} \int_0^{\frac{1}{2}} \frac{ts_1 \sin\frac{t(z-s_1)}{2}\cos\frac{t(z+s_1)}{2}}{t^2} q(t)\,dt,$$

$$I_3(z) = -2\sqrt{\frac{2}{\pi}} \int_0^{\frac{1}{2}} \frac{\sin\frac{t(z-s_1)}{2}\sin\frac{t(z+s_1)}{2}}{t^2} q(t)\,dt.$$

Then (here and so on by C_1, C_2, … we denote arbitrary positive constants) for $z \in \mathbb{C}$

$$|I_1(z)| \le C_1(1+|z|)\Big(\frac{e^{|\operatorname{Im} z|}}{1+|\operatorname{Im} z|} + |\operatorname{Re} z|\Big)^{\frac{1}{2}}, \quad |I_2(z)| \le C_2\Big(\frac{e^{|\operatorname{Im} z|}}{1+|\operatorname{Im} z|} + e^{|\operatorname{Im} z|/2}|\operatorname{Re} z|\Big)^{\frac{1}{2}},$$

$$|I_3(z)| \le C_3\Big(\Big(e^{|\operatorname{Im} z|/2} + (\operatorname{Re} z)^2\Big)\Big(\frac{e^{|\operatorname{Im} z|/2}}{1+|\operatorname{Im} z|} + (\operatorname{Re} z)^2\Big)\Big)^{\frac{1}{2}}.$$

Proof. Since $|\sin tz| = (\operatorname{sh}^2(t\operatorname{Im} z) + \sin^2(t\operatorname{Re} z))^{1/2}$, using Schwartz's inequality, we get

$$|I_1(z)| \le C_1(1+|z|)\|q\|\Big(\int_0^{\frac{1}{2}} \frac{\operatorname{sh}^2(t\operatorname{Im} z) + \sin^2(t\operatorname{Re} z)}{t^2}\,dt\Big)^{\frac{1}{2}} \le$$

$$\le C_2(1+|z|)\Big(\int_0^{\frac{1}{2}} \frac{\operatorname{sh}^2(t\operatorname{Im} z)}{t^2}\,dt + \int_0^{\frac{1}{2}} \frac{\sin^2(t\operatorname{Re} z)}{t^2}\,dt\Big)^{\frac{1}{2}} \le$$

$$\le C_2(1+|z|)\Big(|\operatorname{Im} z| \int_0^{|\operatorname{Im} z|/2} \frac{\operatorname{sh}^2\tau}{\tau^2}\,d\tau + |\operatorname{Re} z| \int_0^{|\operatorname{Re} z|/2} \frac{\sin^2\tau}{\tau^2}\,d\tau\Big)^{\frac{1}{2}} \le$$

$$\le C_3(1+|z|)\Big(|\operatorname{Im} z| \int_0^{|\operatorname{Im} z|/2} \frac{\operatorname{sh}^2\tau}{\tau^2}\,d\tau + |\operatorname{Re} z|\Big)^{\frac{1}{2}} \le C_4(1+|z|)\Big(\frac{e^{|\operatorname{Im} z|}}{1+|\operatorname{Im} z|} + |\operatorname{Re} z|\Big)^{\frac{1}{2}}$$

for $z \in \mathbb{C}$, whence it follows the first inequality of the lemma. Analogously, for $z \in \mathbb{C}$

$$|I_2(z)| \le C_5\|q\|\Big(\int_0^{\frac{1}{2}} \frac{\Big|\sin\frac{t(z-s_1)}{2}\Big|^2\Big|\cos\frac{t(z+s_1)}{2}\Big|^2}{t^2}\,dt\Big)^{\frac{1}{2}} \le C_6\Big(e^{|\operatorname{Im} z|/2}\int_0^{\frac{1}{2}} \frac{\Big|\sin\frac{t(z-s_1)}{2}\Big|^2}{t^2}\,dt\Big)^{\frac{1}{2}} \le$$

$$\le C_6\Big(e^{|\operatorname{Im} z|/2}\int_0^{\frac{1}{2}} \frac{\operatorname{sh}^2\Big(t\operatorname{Im}\frac{z-s_1}{2}\Big) + \sin^2\Big(t\operatorname{Re}\frac{z-s_1}{2}\Big)}{t^2}\,dt\Big)^{\frac{1}{2}} \le$$

$$\le C_7\Big(e^{|\operatorname{Im} z|/2}\Big(\frac{e^{|\operatorname{Im} z|/2}}{1+|\operatorname{Im} z|} + |\operatorname{Re} z|\Big)\Big)^{\frac{1}{2}},$$

and the second inequality of the lemma is proved. Finally, since $\operatorname{sh}^2(t\operatorname{Im} z) \le t^2\operatorname{sh}^2(|\operatorname{Im} z|)$ and $\Big|\sin\operatorname{Re}\frac{t(z-s_1)}{2}\Big| \le \Big|\operatorname{Re}\frac{t(z-s_1)}{2}\Big|$ for any $z \in \mathbb{C}$ and $t \in \mathbb{R}$, we obtain

$$|I_3(z)| \le C_8\|q\|\Big(\int_0^{\frac{1}{2}} \frac{\Big|\sin\frac{t(z-s_1)}{2}\Big|^2\Big|\sin\frac{t(z+s_1)}{2}\Big|^2}{t^4}\,dt\Big)^{\frac{1}{2}} \le$$

$$\le C_9\Big(\int_0^{\frac{1}{2}} \frac{\Big(\operatorname{sh}^2\Big(t\Big|\operatorname{Im}\frac{z-s_1}{2}\Big|\Big) + \sin^2\Big(t\operatorname{Re}\frac{z-s_1}{2}\Big)\Big)\Big(\operatorname{sh}^2\Big(t\Big|\operatorname{Im}\frac{z+s_1}{2}\Big|\Big) + \sin^2\Big(t\operatorname{Re}\frac{z+s_1}{2}\Big)\Big)}{t^4}\,dt\Big)^{\frac{1}{2}} \le$$

$$\le C_{10}\Big(\int_0^{\frac{1}{2}} \frac{\Big(t^2\operatorname{sh}^2\Big(\Big|\operatorname{Im}\frac{z-s_1}{2}\Big|\Big) + \Big(t\operatorname{Re}\frac{z-s_1}{2}\Big)^2\Big)\Big(\operatorname{sh}^2\Big(t\Big|\operatorname{Im}\frac{z+s_1}{2}\Big|\Big) + \sin^2\Big(t\operatorname{Re}\frac{z+s_1}{2}\Big)\Big)}{t^4}\,dt\Big)^{\frac{1}{2}} \le$$

$$\le C_{10}\Big(\Big(\operatorname{sh}^2\Big(\Big|\operatorname{Im}\frac{z-s_1}{2}\Big|\Big) + \Big(\operatorname{Re}\frac{z-s_1}{2}\Big)^2\Big)\int_0^{\frac{1}{2}}\Big(\frac{\operatorname{sh}^2\Big(t\Big|\operatorname{Im}\frac{z+s_1}{2}\Big|\Big)}{t^2} + \Big(\operatorname{Re}\frac{z+s_1}{2}\Big)^2\Big)\,dt\Big)^{\frac{1}{2}} \le$$

$$\le C_{11}\Big(\Big(e^{|\operatorname{Im} z|/2} + (\operatorname{Re} z)^2\Big)\Big(\frac{e^{|\operatorname{Im} z|/2}}{1+|\operatorname{Im} z|} + (\operatorname{Re} z)^2\Big)\Big)^{\frac{1}{2}}, \quad z \in \mathbb{C}.$$

Lemma 1 is proved. □

Lemma 2. *Let $s_1 \in \mathbb{C}$ be an arbitrary number and let an entire function $Q \in E_{2,+}(s_1)$ be defined by formula (3). Then for all $z \in \mathbb{C}$, we have*

$$|Q(z)| \le C_1(1+|z|)\Big(\frac{e^{|\operatorname{Im} z|}}{1+|\operatorname{Im} z|}+|\operatorname{Re} z|\Big)^{\frac12}+C_2\Big(\frac{e^{|\operatorname{Im} z|}}{1+|\operatorname{Im} z|}+e^{|\operatorname{Im} z|/2}|\operatorname{Re} z|\Big)^{\frac12}+$$
$$+C_3\Big(\Big(e^{|\operatorname{Im} z|/2}+(\operatorname{Re} z)^2\Big)\Big(\frac{e^{|\operatorname{Im} z|/2}}{1+|\operatorname{Im} z|}+(\operatorname{Re} z)^2\Big)\Big)^{\frac12}+C_4(1+|z|)\frac{e^{|\operatorname{Im} z|}}{\sqrt{1+|\operatorname{Im} z|}}+C_5.$$

Proof. Indeed, let $I_4(z)=\sqrt{\frac{2}{\pi}}\int_{\frac12}^{1}\frac{q(t)}{t^2}\cos(tz)\,dt,\quad I_5(z)=\sqrt{\frac{2}{\pi}}\int_{\frac12}^{1}\frac{q(t)}{t}\sin(tz)\,dt,$

$$I_6(z)=-\sqrt{\frac{2}{\pi}}\int_{\frac12}^{1}\frac{\cos(ts_1)+ts_1\sin(ts_1)}{t^2}q(t)\,dt.$$

Then $Q(z)=I_1(z)+I_2(z)+I_3(z)+I_4(z)+zI_5(z)+I_6(z)$. According to the Paley-Wiener theorem, the functions $I_4(z)$ and $I_5(z)$ belong to the space PW_1^2,

$$I_4(z)=\sqrt{\frac{2}{\pi}}\int_{\frac12}^{1}e^{itz}\frac{q(t)}{2t^2}\,dt+\sqrt{\frac{2}{\pi}}\int_{-1}^{-\frac12}e^{itz}\frac{q(-t)}{2t^2}\,dt,$$

and applying Schwartz's inequality, we get $|I_4(z)|\le C_{12}\frac{e^{|\operatorname{Im} z|}}{\sqrt{1+|\operatorname{Im} z|}},\quad z\in\mathbb{C}$. Similarly, $|I_5(z)|\le C_{13}\frac{e^{|\operatorname{Im} z|}}{\sqrt{1+|\operatorname{Im} z|}},\quad z\in\mathbb{C}$. Thus, basing on Lemma 1, we obtain the required proposition. Lemma 2 is proved. □

Theorem 3. *Let $(s_k\colon k\in\mathbb{N})$ be a sequence of distinct nonzero complex numbers such that $s_k^2\ne s_m^2$ if $k\ne m$, and let a sequence $(s_k\colon k\in\mathbb{Z}\setminus\{0\})$, $s_{-k}:=-s_k$, be a sequence of zeros of the some even entire function D of finite formal exponential type, for which on the rays $\{z\colon \arg z=\varphi_j\}$, $j\in\{1;2;3;4\}$, $\varphi_1\in[0;\pi/2)$, $\varphi_2\in[\pi/2;\pi)$, $\varphi_3\in(\pi;3\pi/2]$, $\varphi_4\in(3\pi/2;2\pi)$, we have*

$$|D(z)|\ge C(1+|z|)\exp(|\operatorname{Im} z|),$$

where C is a some positive constant. Then the system $(U_k\colon k\in\mathbb{N}\setminus\{1\})$ is complete in the space $L^2((0;1);x^2dx)$.

Proof. Assume the converse. Then, according to Theorem 2, there exists an entire function $Q\in E_{2,+}(s_1)$ for which the sequence $(s_k\colon k\in\mathbb{Z}\setminus\{0\})$ is a subsequence of zeros. Let $T(z)=Q(z)/D(z)$. Then T is an even entire function of finite exponential type, for which (see Lemma 2)

$$|T(z)|\le C_{14}\frac{1}{\sqrt{1+|\operatorname{Im} z|}},\quad \arg z=\varphi_j,\quad j\in\{1,2,3,4\}.$$

Hence, according to the Phragmén-Lindelöf theorem, $T(z)\equiv 0$. Therefore, $Q(z)\equiv 0$. This contradiction concludes the proof. □

Theorem 4. *Let $(s_k\colon k\in\mathbb{Z}\setminus\{0\})$, $s_{-k}:=-s_k$, be a sequence of zeros of the function $J_{-3/2}$. Then the system*

$$(g_k\colon k\in\mathbb{N}\setminus\{1\}),\quad \overline{g}_k(t)=\frac{\pi(1+s_k^2)}{s_k^4t^2}(s_k\sqrt{ts_k}J_{-3/2}(ts_k)-s_1\sqrt{ts_1}J_{-3/2}(ts_1)),$$

is complete in the space $L^2((0;1);x^2dx)$.

Proof. Indeed, the sequence $(s_k\colon k \in \mathbb{Z} \setminus \{0\})$ is a sequence of zeros of the entire function $D(z) = \cos z + z \sin z$, and this function satisfies the conditions of Theorem 3. Therefore, the system $(U_k\colon k \in \mathbb{N} \setminus \{1\})$ is complete in the space $L^2((0;1); x^2 dx)$, and, hence, the system $(g_k\colon k \in \mathbb{N} \setminus \{1\})$ is complete in this space. Theorem 4 is proved. □

Theorem 4 gives a positive answer to the question, formulated at the beginning of this paper.

Lemma 3. *If an even entire function L belongs to the space $E_{2,+}(s_1)$ and has a root at a point $s \neq s_1$ or has multiple root at $s = s_1$, then the function $\widetilde{L}(z) = L(z)/(z^2 - s^2)$ also belongs to $E_{2,+}(s_1)$.*

Proof. Indeed, $\widetilde{L}(s_1) = 0$, the function $\widetilde{L}$ is an even entire function of formal exponential type $\sigma \leq 1$, $\widetilde{L}'(z) = \dfrac{L'(z)(z^2 - s^2) - 2zL(z)}{(z^2 - s^2)^2}$ and $\widetilde{L}'(0) = 0$. Besides,

$$\frac{\widetilde{L}'(z)}{z} = \frac{L'(z)}{z(z^2 - s^2)} - \frac{2L(z)}{(z^2 - s^2)^2}, \quad \int\limits_{1+\operatorname{Re} s}^{+\infty} \left|\frac{L'(x)}{x(x^2 - s^2)}\right|^2 dx \leq C_{15} \int\limits_{1+\operatorname{Re} s}^{+\infty} \left|\frac{L'(x)}{x}\right|^2 dx < +\infty,$$

and according to Lemma 2

$$\int\limits_{1+\operatorname{Re} s}^{+\infty} \left|\frac{L(x)}{(x^2 - s^2)^2}\right|^2 dx \leq C_{16} \int\limits_{1+\operatorname{Re} s}^{+\infty} \left|\frac{(1+|x|)^4}{(x^2 - s^2)^4}\right| dx < +\infty.$$

Hence, the function $\widetilde{L}'(z)/z$ belongs to $L^2(\mathbb{R})$. This completes the proof of the lemma. □

Lemma 4. *If an even entire function L has zeros at points s_k, $k \in \mathbb{N}$ and the function $L(z)/(z^2 - s_2^2)$ belongs to the space $E_{2,+}(s_1)$, then the functions $L_k(z) = L(z)/(z^2 - s_k^2)$ also belong to $E_{2,+}(s_1)$ for every $k \in \mathbb{N} \setminus \{1\}$.*

Proof. Indeed, let $Q_k(z) = (s_k^2 - s_2^2)\dfrac{L(z)}{(z^2 - s_k^2)(z^2 - s_2^2)}$. Then $Q_k(z) = (s_k^2 - s_2^2)\dfrac{L_2(z)}{(z^2 - s_k^2)}$ and $L_k = Q_k + L_2$. Therefore, taking into account the previous lemma, we obtain the required proposition. Lemma 4 is proved. □

A system $(e_k\colon k \in \mathbb{N}_0)$ of the Hilbert space is said to be minimal if for each $n \in \mathbb{N}_0$ the element e_n does not belong to the closure of the linear span of the system $(e_k\colon k \in \mathbb{N}_0 \setminus \{n\})$. A system is minimal if and only if it has a biorthogonal system. A complete system has, at most, one biorthogonal system ([15]).

Theorem 5. *Let $(s_k\colon k \in \mathbb{N})$ be an arbitrary sequence of distinct complex numbers such that $s_k^2 \neq s_m^2$ if $k \neq m$. If the sequence $(s_k\colon k \in \mathbb{N})$ is a subsequence of zeros of some even entire function W which has simple roots at all points s_k and the function $W(z)/(z^2 - s_2^2)$ belongs to $E_{2,+}(s_1)$, then the system $(U_k\colon k \in \mathbb{N} \setminus \{1\})$ has in the space $L^2((0;1); x^2 dx)$ a biorthogonal system $(\gamma_k\colon k \in \mathbb{N} \setminus \{1\})$. The biorthogonal system $(\gamma_k\colon k \in \mathbb{N} \setminus \{1\})$ is formed, in particular, by the functions γ_k, defined by the equality*

$$\overline{\gamma}_k(t) = -\sqrt{\frac{2}{\pi}} \int_0^{+\infty} \frac{W_k'(z)}{tz} \cos(tz)\, dz, \quad W_k(z) := \frac{2s_k(s_1^2 - s_k^2)W(z)}{W'(s_k)(z^2 - s_k^2)}. \tag{4}$$

Proof. In fact, according to Lemma 4, the functions W_k belong to the space $E_{2,+}(s_1)$. Therefore, there exist nonzero elements γ_k of the space $L^2((0;1);x^2dx)$ such that

$$W_k(z) = \int_0^1 (z\sqrt{tz}J_{-3/2}(tz) - s_1\sqrt{ts_1}J_{-3/2}(ts_1))\gamma_k(t)\,dt,$$

and by Theorem 1 the functions γ_k can be found on written formulas. Moreover, $\dfrac{W_k(s_n)}{s_1^2 - s_n^2} = \begin{cases} 1, n = k, \\ 0, n \neq k, \end{cases}$ and we obtain the required proposition. Theorem 5 is proved. □

Theorem 6. *Let $(s_k\colon k \in \mathbb{N})$ be an arbitrary sequence of complex numbers such that $s_k^2 \neq s_m^2$ as $k \neq m$. The system $(U_k\colon k \in \mathbb{N} \setminus \{1\})$ is complete and minimal in the space $L^2((0;1);x^2dx)$ if and only if the sequence $(s_k\colon k \in \mathbb{Z} \setminus \{0\})$, $s_{-k}{:=}-s_k$, is a subsequence of zeros of some even entire function W such that the function $W(z)/(z^2 - s_2^2)$ belongs to the space $E_{2,+}(s_1)$ and the function W does not belong to this space.*

Proof. If the considered system is minimal then there exists a nonzero function $\gamma_2 \in$ $\in L^2((0;1);x^2dx)$ such that

$$\int_0^1 (s_k\sqrt{ts_k}J_{-3/2}(ts_k) - s_1\sqrt{ts_1}J_{-3/2}(ts_1))\gamma_2(t)\,dt = \begin{cases} 1, k = 2, \\ 0, k \neq 2. \end{cases}$$

Let $T(z) = \int_0^1 (z\sqrt{tz}J_{-3/2}(tz) - s_1\sqrt{ts_1}J_{-3/2}(ts_1))\gamma_2(t)\,dt$. The function $W(z) = (z^2 - s_2^2)T(z)$ is the required, because the function $T(z) = W(z)/(z^2 - s_2^2)$ belongs to the space $E_{2,+}(s_1)$ and has zeros at all points s_k, all its zeros are simple and it has no other zeros. Indeed, if $\widetilde{s}$ is another root of the function W, then the function $\widetilde{G}(z) = W(z)/(z^2 - \widetilde{s}^2)$ which has roots at all points s_k, would belongs to the space $E_{2,+}(s_1)$ that, according to Theorem 2, contradicts the completeness of the considered system. Besides, the function W does not belong to $E_{2,+}(s_1)$, because otherwise the system would be incomplete. Conversely, if all the conditions of the theorem hold then, basing on Theorem 5, we obtain the required proposition. The proof of Theorem 6 is thus completed. □

Corollary 3. *Let $(s_k\colon k \in \mathbb{Z} \setminus \{0\})$, $s_{-k}{:=}-s_k$, be a sequence of zeros of the function $J_{-3/2}$. Then the system $(U_k\colon k \in \mathbb{N} \setminus \{1\})$ has in the space $L^2((0;1);x^2dx)$ a biorthogonal system $(\gamma_k\colon k \in \mathbb{N} \setminus \{1\})$ which formed by the functions γ_k, defined by the formula*

$$\overline{\gamma}_k(t) = \frac{\pi(1 + s_k^2)(s_1^2 - s_k^2)}{s_k^3}\sqrt{ts_k}J_{-3/2}(ts_k).$$

Proof. Indeed, the sequence $(s_k\colon k \in \mathbb{Z} \setminus \{0\})$ is a sequence of zeros of even entire function $W(z) = \cos z + z\sin z$, that satisfy the conditions of Theorem 5. Then, according to this theorem, the system $(U_k\colon k \in \mathbb{N}\setminus\{1\})$ has in the space $L^2((0;1);x^2dx)$ a biorthogonal system $(\gamma_k\colon k \in \mathbb{N} \setminus \{1\})$ which formed by the functions γ_k, defined by equality (4), where

$$W_k(z){:=}\frac{2(s_1^2 - s_k^2)(\cos z + z\sin z)}{(z^2 - s_k^2)\cos s_k}.$$

Therefore,

$$\overline{\gamma}_k(t) = -\sqrt{\frac{2}{\pi}}\frac{1}{t}\int_0^{+\infty} (W_k(z) - W_k(0))\frac{\cos(tz) + tz\sin(tz)}{z^2}\,dz =$$

$$= -2\sqrt{\frac{2}{\pi}}\frac{s_1^2 - s_k^2}{t\cos s_k}\int_0^{+\infty} \frac{(\cos z + z\sin z - 1)(\cos(tz) + tz\sin(tz))}{z^2(z^2 - s_k^2)}\,dz =$$

$$= -2\sqrt{\frac{2}{\pi}}\frac{s_1^2 - s_k^2}{ts_k^2 \cos s_k}\int_0^{+\infty} \frac{\cos(tz) + tz\sin(tz)}{z^2 - s_k^2}\,dz.$$

Let $\eta(z;t) = tz^2 e^{i(1-t)z} - tz^2 e^{i(1+t)z} - ize^{i(1-t)z} - ize^{i(1+t)z} + itze^{i(1-t)z} - itze^{i(1+t)z} + e^{i(1-t)z} + e^{i(1+t)z} + 2itze^{itz} - 2e^{itz}$. Then $(\cos z + z\sin z - 1)(\cos(tz) + tz\sin(tz)) = \dfrac{(\eta(z;t) + \eta(-z;t))}{4}$. Hence,

$$\overline{\gamma}_k(t) = -\frac{1}{\sqrt{2\pi}}\frac{s_1^2 - s_k^2}{t\cos s_k}\int_{-\infty}^{+\infty}\frac{\eta(z;t)}{z^2(z^2 - s_k^2)}\,dz - \sqrt{\frac{2}{\pi}}\frac{s_1^2 - s_k^2}{ts_k^2\cos s_k}\,\mathrm{Re}\int_{-\infty}^{+\infty}\frac{e^{itz}}{z^2 - s_k^2}\,dz -$$

$$-\sqrt{\frac{2}{\pi}}\frac{s_1^2 - s_k^2}{s_k^2\cos s_k}\,\mathrm{Im}\int_{-\infty}^{+\infty}\frac{ze^{itz}}{z^2 - s_k^2}\,dz =$$

$$= \frac{\sqrt{2\pi}(s_1^2 - s_k^2)}{ts_k^3\cos s_k}(\sin s_k - s_k\cos s_k)(\cos(ts_k) + ts_k\sin(ts_k)) = \frac{\pi(1 + s_k^2)(s_1^2 - s_k^2)}{s_k^3}\sqrt{ts_k}\,J_{-3/2}(ts_k).$$

Corollary 3 is proved. □

Remark 1. *Corollary 3 also follows from Theorem A.*

REFERENCES

1. Ватсон Г.Н. Теория Бесселевых функций. – М.: ИЛ, 1949. – Т.1. – 787 с.
2. Бейтмен Г., Эрдейи А. Высшие трансцендентные функции. – М.: Наука, 1974. – Т.2. – 296 с.
3. Лебедев Н.Н. Специальные функции и их приложения. – М.–Л: Физматгиз, 1963. – 359 с.
4. Boas R.P., Pollard H. *Complete sets of Bessel and Legendre functions*// Annals of Math. – 1947. – V.48, №2. – P. 366–384.
5. Винницький Б.В., Шавала О.В. *Обмеженість розв'язків лінійного диференціального рівняння другого порядку і одна крайова задача для рівняння Бесселя*// Mat. Stud. – 2008. – Т.30, №1. – С. 31–41.
6. Vynnyts'kyi B.V., Shavala O.V. *On completeness of the system* $\{\cos(\rho_n x) + \rho_n x\sin(\rho_n x)\}$ *and a boundary value problem for Bessel operator*// Int. Conf. "Analysis and topology": Abstracts. Lviv. – 2008. – P. 54–55.
7. Шавала О.В. Деякі властивості лінійних диференціальних рівнянь другого порядку з мероморфними коефіцієнтами: Дис. ... канд. фіз.-мат. наук. – Дрогобич. – 2008. – 127 с.
8. Vynnyts'kyi B.V., Dilnyi V.M. *On some analogues of Paley-Wiener theorem and one boundary value problem for Bessel operator*// Int. Conf. on complex analysis in memory of A.A. Gol'dberg: Abstracts. Lviv. – 2010. – P. 63–64.
9. Vynnyts'kyi B.V., Khats' R.V. *Approximation properties of Bessel functions and a generalization of the notion of eigenvector*// Int. Conf. on complex analysis in memory of A.A. Gol'dberg: Abstracts. Lviv. – 2010. – P. 64–65.
10. Кусис П. Введение в теорию пространств H^p. – М.: Мир, 1984. – 368 с.
11. Гарнетт Дж. Ограниченные аналитические функции. – М.: Мир, 1984. – 472 с.
12. Левин Б.Я. Распределение корней целых функций. – М.: Гостехиздат, 1956. – 632 с.
13. Levin B.Ya. Lectures on entire functions. Transl. Math. Monographs, V.150. Amer. Math. Soc., Providence RI, 1996. – 248 p.
14. Седлецкий А. М. Классы аналитических преобразований Фурье и экспоненциальные аппроксимации. – М.: Физматлит, 2005. – 504 с.
15. Singer I. Bases in Banach Spaces. – Berlin: Springer-Verlag, 1970. – V.1. – 672 p.

Drohobych Ivan Franko State Pedagogical University,
Institute of Physics, Mathematics and Informatics
vynnytskyi@ukr.net; khats@ukr.net

Received 15.06.2010