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REGULARITY OF GROWTH OF FOURIER COEFFICIENTS OF ENTIRE FUNCTIONS OF IMPROVED REGULAR GROWTH

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We establish a criterion for the improved regular growth of entire functions of positive order with zeros on a finite system of half-lines in terms of their Fourier coefficients.

The theory of entire functions of completely regular growth in the Levin–Pfluger sense [1] establishes a relationship between the regularity of growth of an entire function and the regular behavior of the sequence of its zeros. Numerous investigations have been devoted to the development of the Levin–Pfluger theory and generalization of its results to other classes of functions (see [2, 3]). At present, many different conditions are known that are necessary and sufficient for the completely regular growth of entire functions. In particular, a criterion for the completely regular growth of entire functions of positive order in terms of their Fourier coefficients was established in [4].

Theorem A [4]. *For an entire function f of order $\rho \in (0, +\infty)$ to be a function of completely regular growth, it is necessary and sufficient that the following limits exist for all $k \in \mathbb{Z}$:*

$$\lim_{r\to+\infty} \frac{c_k(r, \log|f|)}{r^\rho} = c_k, \qquad c_k(r, \log|f|) := \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\varphi} \log|f(re^{i\varphi})|\, d\varphi.$$

In [5, p. 76] (see also [8]), an analog of Theorem A was obtained in the class of meromorphic functions of finite λ type of completely regular growth. In [6], an analog of Theorem A was obtained for the class of entire functions of strongly regular growth of zero order with zeros on a finite system of half-lines. Criteria for the completely regular growth of entire and meromorphic functions of positive order in the metric of $L^p[0, 2\pi]$ were established in [7] and [5, p. 78; 8, 9], respectively (see also [10]).

In [11, 12] (see also [13]), the notion of entire function of improved regular growth was introduced, and a criterion for this regularity was obtained in terms of the distribution of zeros under the condition that they are located on a finite system of half-lines. In [14], this notion was generalized to subharmonic functions. An entire function f is called a function of *improved regular growth* [11] if, for certain $\rho \in (0, +\infty)$ and $\rho_1 \in (0, \rho)$ and a 2π-periodic ρ-trigonometrically convex function $h \not\equiv -\infty$, there exists a set $U \subset \mathbb{C}$ contained in the union of disks with finite sum of radii and such that

$$\log|f(z)| = |z|^\rho h(\arg z) + o(|z|^{\rho_1}), \quad U \not\ni z \to \infty.$$

If an entire function f is a function of improved regular growth, then it has the order ρ and indicator h [11].

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Let f be an entire function, let $f(0) = 1$, let (λ_n) be the sequence of its zeros, let p be the least nonnegative integer number for which

$$\sum_{n\in\mathbb{N}} |\lambda_n|^{-p-1} < +\infty,$$

let

$$n(r,\psi; f) := \sum_{\substack{|\lambda_n|\le r\\ \arg\lambda_n=\psi}} 1,$$

and let Q_ρ be the coefficient of z^ρ in the exponential factor in the Hadamard–Borel representation [1, p. 38] of an entire function f of order $\rho \in (0,+\infty)$.

Theorem B [11]. *An entire function f of noninteger order $\rho \in (0,+\infty)$ with zeros on a finite system of half-lines $\{z\colon \arg z = \psi_j\}$, $j \in \{1,\dots,m\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$, is a function of improved regular growth if and only if, for a certain $\rho_2 \in (0,\rho)$ and each $j \in \{1,\dots,m\}$, one has*

$$n(t,\psi_j; f) = \Delta_j t^\rho + o(t^{\rho_2}), \qquad t \to +\infty, \quad \Delta_j \in [0,+\infty). \tag{1}$$

In this case,

$$h(\varphi) = \sum_{j=1}^{m} h_j(\varphi), \tag{2}$$

where $h_j(\varphi)$ is the 2π-periodic function defined on the interval $[\psi_j, \psi_j + 2\pi)$ by the equality

$$h_j(\varphi) = \frac{\pi\Delta_j}{\sin\pi\rho}\cos\rho(\varphi - \psi_j - \pi).$$

Theorem C [12]. *An entire function f of order $\rho \in \mathbb{N}$ with zeros on a finite system of half-lines $\{z\colon \arg z = \psi_j\}$, $j \in \{1,\dots,m\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$, is a function of improved regular growth if and only if equality (1) holds for a certain $\rho_2 \in (0,\rho)$ and each $j \in \{1,\dots,m\}$ and, for certain $\rho_3 \in (0,\rho)$ and $\delta_f \in \mathbb{C}$, one has*

$$\sum_{0<|\lambda_n|\le r} \lambda_n^{-\rho} = \delta_f + o(r^{\rho_3-\rho}), \quad r \to +\infty.$$

In this case,

$$h(\varphi) = \begin{cases} \tau_f \cos(\rho\varphi + \theta_f) + \displaystyle\sum_{j=1}^{m} h_j(\varphi), & \rho = p, \\ Q_\rho \cos\rho\varphi, & \rho = p+1, \end{cases} \tag{3}$$

where $\tau_f = |\delta_f/\rho + Q_\rho|$, $\theta_f = \arg(\delta_f/\rho + Q_\rho)$, and $h_j(\varphi)$ is the 2π-periodic function defined on the interval $[\psi_j, \psi_j + 2\pi)$ by the equality

$$h_j(\varphi) = \Delta_j(\pi - \varphi + \psi_j)\sin\rho(\varphi - \psi_j) - \frac{\Delta_j}{\rho}\cos\rho(\varphi - \psi_j).$$

The aim of the present paper is to establish an analog of the aforementioned results of [4, 6] for the class of entire functions of improved regular growth with zeros on a finite system of half-lines.

Theorem 1. *An entire function* f *of order* $\rho \in (0, +\infty)$ *with zeros on a finite system of half-lines* $\{z : \arg z = \psi_j\}$, $j \in \{1, \ldots, m\}$, $0 \le \psi_1 < \psi_2 < \ldots < \psi_m < 2\pi$, *is a function of improved regular growth if and only if, for certain* $\rho_4 \in (0, \rho)$ *and* $k_0 \in \mathbb{Z}$ *and each* $k \in \{k_0, k_0 + 1, \ldots, k_0 + m - 1\}$, *one has*

$$c_k(r, \log|f|) = c_k r^{\rho} + o(r^{\rho_4}), \quad r \to +\infty. \tag{4}$$

In the proof of Theorem 1, we use the following auxiliary statements:

Lemma 1 [11]. *Let* $\rho \in (0, +\infty)$, *let* $\rho_1 \in (0, \rho)$, *and let* f *be an entire function of improved regular growth. Then there exists a sequence* (r_s) *such that* $0 < r_s \uparrow +\infty$, $r_{s+1}^{\rho} - r_s^{\rho} = o(r_s^{\rho_1})$, *and* $\log|f(r_s e^{i\varphi})| = r_s^{\rho} h(\varphi) + o(r_s^{\rho_1})$, $s \to +\infty$, *uniformly in* $\varphi \in [0, 2\pi]$.

Lemma 2 [15]. *Let* f *be an entire function of order* $\rho \in (0, +\infty)$ *with indicator* $h(\varphi)$. *If the sequence* (r_s) *from Lemma 1 exists, then there exists* $\rho_4 \in (0, \rho)$ *such that relation (4) holds for each* $k \in \mathbb{Z}$.

Let [5, p. 104]

$$n_k(r, f) := \sum_{|\lambda_n| \le r} e^{-ik \arg \lambda_n}, \qquad N_k(r, f) := \int_0^r \frac{n_k(t, f)}{t}\, dt, \quad k \in \mathbb{Z}.$$

It is known [5, p. 107] that

$$N_k(r, f) = c_k(r, \log|f|) - k^2 \int_0^r \frac{dt}{t} \int_0^t \frac{c_k(u, \log|f|)}{u}\, du, \qquad k \in \mathbb{Z}, \quad r > 0. \tag{5}$$

Lemma 3. *Let* f *be an entire function of improved regular growth of order* $\rho \in (0, +\infty)$. *Then, for a certain* $\rho_4 \in (0, \rho)$ *and each* $k \in \mathbb{Z}$, *relation (4) is true and*

$$N_k(r, f) = c_k(1 - k^2/\rho^2) r^{\rho} + o(r^{\rho_4}), \quad r \to +\infty. \tag{6}$$

Proof. Indeed, relation (4) follows directly from Lemmas 1 and 2. Let us prove relation (6). If (4) is true, then, using (5) and passing to the limit as $r \to +\infty$, we obtain

$$N_k(r, f) = c_k r^{\rho} + o(r^{\rho_4}) - k^2 \int_0^r \frac{dt}{t} \int_0^t (c_k u^{\rho-1} + o(u^{\rho_4 - 1}))\, du = c_k(1 - k^2/\rho^2) r^{\rho} + o(r^{\rho_4}).$$

Lemma 3 is proved.

Lemma 3 yields the following statement:

Lemma 4. *If f is an entire function of improved regular growth of order $\rho \in (0, +\infty)$ with zeros on a finite system of half-lines $\{z \colon \arg z = \psi_j\}$, $j \in \{1, \ldots, m\}$, $0 \le \psi_1 < \psi_2 < \ldots < \psi_m < 2\pi$, then, for a certain $\rho_4 \in (0, \rho)$ and each $k \in \mathbb{Z}$, relations (4) and (6) are true, where*

$$c_k := \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\varphi} h(\varphi)\, d\varphi = \frac{\rho}{\rho^2 - k^2} \sum_{j=1}^m \Delta_j e^{-ik\psi_j}, \quad \Delta_j \in [0, +\infty), \tag{7}$$

if ρ is a noninteger number, and

$$c_k = \begin{cases} \dfrac{\rho}{\rho^2 - k^2} \displaystyle\sum_{j=1}^m \Delta_j e^{-ik\psi_j}, & |k| \ne \rho = p, \\[2ex] \dfrac{\tau_f e^{i\theta_f}}{2} - \dfrac{1}{4\rho} \displaystyle\sum_{j=1}^m \Delta_j e^{-i\rho\psi_j}, & k = \rho = p, \\[2ex] 0, & |k| \ne \rho = p + 1, \\[2ex] \dfrac{Q_\rho}{2}, & k = \rho = p + 1, \end{cases} \tag{8}$$

if $\rho \in \mathbb{N}$.

Proof. Indeed, using (2), we get

$$c_k = \frac{1}{2\pi} \sum_{j=1}^m \frac{\pi \Delta_j}{\sin \pi\rho} \left(\int_{\psi_j}^{\psi_j + 2\pi} e^{-ik\varphi} \cos \rho(\varphi - \psi_j - \pi)\, d\varphi \right) = \frac{\rho}{\rho^2 - k^2} \sum_{j=1}^m \Delta_j e^{-ik\psi_j}, \quad k \in \mathbb{Z}.$$

By analogy, using (3), we obtain

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\varphi} \tau_f \cos(\rho\varphi + \theta_f)\, d\varphi$$

$$+ \frac{1}{2\pi} \sum_{j=1}^m \left(\int_{\psi_j}^{\psi_j + 2\pi} e^{-ik\varphi} \left(\Delta_j (\pi - \varphi + \psi_j) \sin \rho(\varphi - \psi_j) - \frac{\Delta_j}{\rho} \cos \rho(\varphi - \psi_j) \right) d\varphi \right)$$

$$= \frac{\rho}{\rho^2 - k^2} \sum_{j=1}^m \Delta_j e^{-ik\psi_j}, \quad |k| \ne \rho = p,$$

and

$$c_\rho = \frac{\tau_f e^{i\theta_f}}{2} - \frac{1}{4\rho}\sum_{j=1}^{m} \Delta_j e^{-i\rho\psi_j}.$$

The case $\rho = p + 1$ can be considered by analogy.

Lemma 4 is proved.

Lemma 5. *Let* $\rho \in (0, +\infty)$. *In order that the equality*

$$N(r, \psi_j; f) := \int_0^r \frac{n(t, \psi_j; f)}{t}\, dt = \frac{\Delta_j}{\rho} r^\rho + o(r^{\rho_4}), \quad r \to +\infty, \quad \Delta_j \in [0, +\infty), \tag{9}$$

hold for a certain $\rho_4 \in (0, \rho)$ *and each* $j \in \{1, \ldots, m\}$, *it is necessary and sufficient that, for certain* $\rho_4 \in (0, \rho)$ *and* $k_0 \in \mathbb{Z}$ *and each* $k \in \{k_0, k_0 + 1, \ldots, k_0 + m - 1\}$, *relation (6) with* c_k *defined by (7) and (8) be true.*

Proof. The *necessity* follows from the relations

$$N_k(r, f) = \sum_{j=1}^{m} e^{-ik\psi_j} N(r, \psi_j; f).$$

Let us prove the *sufficiency*. Without loss of generality, we can assume that $k_0 = 0$. Then, by analogy with [5, p. 127; 6], for $k \in \{0, 1, \ldots, m-1\}$ we get

$$N_0(r, f) = N(r, \psi_1; f) + N(r, \psi_2; f) + \ldots + N(r, \psi_m; f),$$

$$N_1(r, f) = e^{-i\psi_1} N(r, \psi_1; f) + e^{-i\psi_2} N(r, \psi_2; f) + \ldots + e^{-i\psi_m} N(r, \psi_m; f),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$N_{m-1}(r, f) = e^{-i(m-1)\psi_1} N(r, \psi_1; f) + e^{-i(m-1)\psi_2} N(r, \psi_2; f) + \ldots + e^{-i(m-1)\psi_m} N(r, \psi_m; f).$$

This is a system of linear equations for the unknowns $N(r, \psi_j; f)$, $j \in \{1, \ldots, m\}$. Its determinant is the nonzero Vandermonde determinant. Therefore, the functions $N(r, \psi_j; f)$, $j \in \{1, \ldots, m\}$, can be represented as linear combinations of the functions $N_k(r, f)$, $k \in \{0, 1, \ldots, m-1\}$. Solving this system by the Cramer rule and using (6), we obtain relation (9).

Lemma 5 is proved.

Lemma 6 [11, 16]**.** *Let* $\rho \in (0, +\infty)$. *For equality (1) to be true for a certain* $\rho_2 \in (0, \rho)$ *and each* $j \in \{1, \ldots, m\}$, *it is necessary and sufficient that relation (9) hold for a certain* $\rho_4 \in (0, \rho)$ *and each* $j \in \{1, \ldots, m\}$.

Remark 1. The necessity parts of Theorems B and C follow from Lemmas 4–6 and Lemma 6 in [12].

Proof of Theorem 1. The *necessity* follows from Lemma 4. The *sufficiency* follows from Lemmas 4–6, Lemma 6 in [12], and the sufficiency parts of Theorems B and C.

Theorem 1 is unimprovable in the sense of the following theorem:

Theorem 2. *For each* $m \in \mathbb{N}\backslash\{1\}$, *there exists an entire function* f *of noninteger order* $\rho \in (0,+\infty)$ *with zeros on a finite system of half-lines* $\{z\colon \arg z = \psi_j\}$, $\psi_j := 2\pi(j-1)/m$, $j \in \{1,\ldots,m\}$, *such that*

$$c_0(r, \log|f|) = \frac{m}{\rho} r^\rho - \frac{m}{\rho^2}\frac{r^\rho}{\log r} + o\left(\frac{r^\rho}{\log r}\right), \quad r \to +\infty,$$

for any $\rho_4 \in (0,\rho)$ *and each* $k \in \{1,\ldots,m-1\}$, *relation (4) is true, and* f *is not a function of improved regular growth.*

Proof. Let $\rho \in (0,+\infty)$ be a noninteger number and let

$$\mu_n = \left(n + \frac{n}{\log n}\right)^{1/\rho},$$

$$\{\lambda_n\colon n \in \mathbb{N}\backslash\{1\}\} := \bigcup_{j=1}^{m} \left\{\mu_n e^{i\frac{2\pi(j-1)}{m}}\colon n \in \mathbb{N}\backslash\{1\}\right\},$$

$$f(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{\lambda_n}\right)\exp\left(\sum_{\nu=1}^{p}\frac{1}{\nu}\left(\frac{z}{\lambda_n}\right)^\nu\right), \quad p = [\rho].$$

Then

$$\mu_n^\rho = n + \frac{n}{\log n} = (1+o(1))n \quad \text{and} \quad n = (1+o(1))\mu_n^\rho \quad \text{as} \quad n \to +\infty.$$

As $n \to +\infty$, we get

$$n = \mu_n^\rho\left(1 + \frac{1}{\log n}\right)^{-1} = \mu_n^\rho\left(1 + \frac{1}{\log((1+o(1))\mu_n^\rho)}\right)^{-1}$$

$$= \mu_n^\rho\left(1 - \frac{1}{\log((1+o(1))\mu_n^\rho)} + o\left(\frac{1}{\log((1+o(1))\mu_n^\rho)}\right)\right)$$

$$= \mu_n^\rho\left(1 - \frac{1}{\rho\log\mu_n + o(1)} + o\left(\frac{1}{\rho\log\mu_n + o(1)}\right)\right)$$

$$= \mu_n^\rho - \frac{\mu_n^\rho}{\rho\log\mu_n} + o\left(\frac{\mu_n^\rho}{\log\mu_n}\right).$$

Further, let $\mu_n \le t < \mu_{n+1}$. Then

$$n\left(t, \frac{2\pi(j-1)}{m}; f\right) = n = \mu_n^\rho - \frac{\mu_n^\rho}{\rho\log\mu_n} + o\left(\frac{\mu_n^\rho}{\log\mu_n}\right) \le t^\rho - \frac{t^\rho}{\rho\log t} + o\left(\frac{t^\rho}{\log t}\right), \quad t \to +\infty.$$

On the other hand,

$$n\left(t, \frac{2\pi(j-1)}{m}; f\right) = n+1-1 = \mu_{n+1}^{\rho} - \frac{\mu_{n+1}^{\rho}}{\rho \log \mu_{n+1}} + o\left(\frac{\mu_{n+1}^{\rho}}{\log \mu_{n+1}}\right) - 1$$

$$\geq t^{\rho} - \frac{t^{\rho}}{\rho \log t} + o\left(\frac{t^{\rho}}{\log t}\right), \quad t \to +\infty.$$

Therefore,

$$n\left(t, \frac{2\pi(j-1)}{m}; f\right) = t^{\rho} - \frac{t^{\rho}}{\rho \log t} + o\left(\frac{t^{\rho}}{\log t}\right) \quad \text{as} \quad t \to +\infty$$

for each $j \in \{1, \ldots, m\}$. Thus, relation (1) is not true for any $\rho_2 \in (0, \rho)$, and, according to Theorems B and C, the entire function f is not a function of improved regular growth. Furthermore, for each $j \in \{1, \ldots, m\}$, we obtain

$$N\left(r, \frac{2\pi(j-1)}{m}; f\right) = \frac{r^{\rho}}{\rho} - \frac{r^{\rho}}{\rho^2 \log r} + o\left(\frac{r^{\rho}}{\log r}\right), \quad r \to +\infty.$$

Therefore,

$$c_0(r, \log|f|) = \sum_{j=1}^{m} N\left(r, \frac{2\pi(j-1)}{m}; f\right) = \frac{m}{\rho} r^{\rho} - \frac{m}{\rho^2} \frac{r^{\rho}}{\log r} + o\left(\frac{r^{\rho}}{\log r}\right), \quad r \to +\infty.$$

Thus, relation (4) is not true for $k = 0$. Moreover (see [15; 13, p. 77]),

$$c_k(r, \log|f|) = \frac{1}{2k} \sum_{j=1}^{m} \sum_{\mu_n \leq r} \left[\left(\frac{r}{\lambda_n}\right)^k - \left(\frac{\overline{\lambda}_n}{r}\right)^k\right]$$

$$= \frac{1}{2k} \sum_{\mu_n \leq r} \left[\left(\frac{r}{\mu_n}\right)^k - \left(\frac{\mu_n}{r}\right)^k\right] \sum_{j=1}^{m} e^{-ik\frac{2\pi(j-1)}{m}}, \quad 1 \leq k \leq p,$$

and

$$c_k(r, \log|f|) = -\frac{1}{2k} \sum_{j=1}^{m} \left\{\sum_{\mu_n > r} \left(\frac{r}{\lambda_n}\right)^k + \sum_{\mu_n \leq r} \left(\frac{\overline{\lambda}_n}{r}\right)^k\right\}$$

$$= -\frac{1}{2k} \left\{\sum_{\mu_n > r} \left(\frac{r}{\mu_n}\right)^k + \sum_{\mu_n \leq r} \left(\frac{\mu_n}{r}\right)^k\right\} \sum_{j=1}^{m} e^{-ik\frac{2\pi(j-1)}{m}}, \quad k \geq p+1.$$

Since

$$\sum_{j=1}^{m} e^{-ik\frac{2\pi(j-1)}{m}} = \frac{1 - e^{-2\pi k i}}{1 - e^{-i\frac{2\pi k}{m}}} = 0, \quad k \in \{1, \ldots, m-1\},$$

we conclude that $c_k(r, \log|f|) = 0$ for each $k \in \{1, \ldots, m-1\}$ and all $r > 0$. Therefore, relation (4) holds for any $\rho_4 \in (0, \rho)$ and each $k \in \{1, \ldots, m-1\}$.

Theorem 2 is proved.

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